Exercise 1

Prove Rudin’s Theorem 7.9: Suppose \( \{f_n\} \) is a sequence of functions, \( f_n : E \to \mathbb{R} \). Suppose \( \lim_{n \to \infty} f_n(x) = f(x) \) for all \( x \in E \), i.e. \( f_n \to f \) pointwise on \( E \). Put \( M_n = \sup \{|f_n(x) - f(x)| : x \in E\} \). Then \( f_n \to f \) uniformly on \( E \) if and only if \( \lim_{n \to \infty} M_n = 0 \).

Exercise 2 (Rudin 7.1)

Suppose \( f_n : E \to \mathbb{R} \) is a sequence of functions. We say that \( \{f_n\} \) is uniformly bounded on \( E \) if there exists some \( M \in \mathbb{R} \) such that for all \( x \in E \) and all \( n \in \mathbb{N} \) we have \( |f_n(x)| < M \).

Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Exercise 3 (Rudin 7.2)

If \( \{f_n\} \) and \( \{g_n\} \) are sequences of functions mapping \( E \to \mathbb{R} \), and converging uniformly on \( E \), prove that \( \{f_n + g_n\} \) converges uniformly on \( E \). If in addition each \( f_n \) is bounded and each \( g_n \) is bounded, prove that \( \{f_n g_n\} \) converges uniformly on \( E \).

Exercise 4 (Rudin 7.3)

Construct sequences \( \{f_n\}, \{g_n\} \) of functions mapping \( X \to \mathbb{R} \) (with \( X \) some metric space), such that \( \{f_n\} \) and \( \{g_n\} \) both converge uniformly on \( X \), but \( \{f_n g_n\} \) does not converge uniformly on \( X \).