In your solutions to these exercises you may freely use any results proven in class or in Rudin chapters 1-2, without reproving them.

**Exercise 1**

Show that the subset of $\mathbb{R}^2$ given by $E = \{(x, y) \in \mathbb{R}^2 \mid x < y\}$ is open.

**Answer of exercise 1**

We need to show that every point $(x, y) \in E$ is an interior point. So, suppose $(x, y) \in E$. Then $x < y$. Let $\epsilon = \frac{1}{2}(y - x)$. Then $\epsilon > 0$. Suppose $(x', y') \in N_r((x, y))$. Then $|x' - x|^2 + |y' - y|^2 < \epsilon^2$. This implies that $|x' - x| < \epsilon$ and $|y' - y| < \epsilon$. But then it follows that $x' < x + \epsilon = x + \frac{1}{2}(y - x) = \frac{1}{2}(x + y)$, and $y' > y - \epsilon = y - \frac{1}{2}(y - x) = \frac{1}{2}(x + y)$. Thus $y' > x'$. In other words, $N_r((x, y)) \subset E$.

**Exercise 2 (Rudin 2.7)**

Let $A_1, A_2, \ldots$ be subsets of a metric space $X$.

1. If $B_n = \bigcup_{i=1}^n A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^n \bar{A}_i$, for any $n \in \mathbb{N}$.

2. If $B = \bigcup_{i=1}^\infty A_i$, prove that $\bar{B} \supset \bigcup_{i=1}^\infty \bar{A}_i$. Show, by an example, that this inclusion can be proper, i.e. it may happen that $\bar{B} \neq \bigcup_{i=1}^\infty \bar{A}_i$.

**Answer of exercise 2**

First we show that more generally, for any collection $\{A_\alpha\}$ of subsets of $X$, and $B = \bigcup_\alpha A_\alpha$, we have $\bar{B} \supset \bigcup_\alpha \bar{A}_\alpha$. First, $A_\alpha \subset B$. Now suppose $x \in A'_\alpha$. Then for any neighborhood $N$ of $x$, there exists some $y \neq x$ with $y \in N \cap A_\alpha$. But then $y \in N \cap B$ also. Hence $x \in B'$. So $A'_\alpha \subset B'$. This gives the desired $\bar{B} \supset \bigcup_\alpha \bar{A}_\alpha$.

Now, we show that for a finite collection $\{A_1, \ldots, A_n\}$ and $B = \bigcup_{i=1}^n A_i$ we have $\bar{B} \subset \bigcup_{i=1}^n \bar{A}_i$. All we need show is that if $x$ is a limit point of $B$ then $x$ is a limit point of some $A_i$. To show this, suppose that $x$ is not a limit point of any $A_i$. In this case there exists some $\epsilon_i$ for which $N_{\epsilon_i}(x)$ does not contain any point of $A_i$ (other than $x$). But then taking $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$, $N_{\epsilon}(x)$ does not contain any point of any $A_i$ (other than $x$), thus it contains no point of $B$ (other than $x$). This contradicts the assumption that $x$ is a limit point of $B$.

These two assertions together establish the statements which were to be proven. The only thing left is to show that it may happen that $\bar{B} \neq \bigcup_{i=1}^\infty \bar{A}_i$. For this, take $A_i = \{1/i\} \subset \mathbb{R}$. Each $\bar{A}_i = A_i$, so $\bigcup_{i=1}^\infty \bar{A}_i = \bigcup_{i=1}^\infty A_i$. However, $\bar{B}$ contains $0$ in addition to the union of the $A_i$. 

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Exercise 3 \((\text{Rudin 2.8})\)

Is every point of every open set \(E \subset \mathbb{R}^2\) a limit point of \(E\)? Answer the same question for closed sets in \(\mathbb{R}^2\).

**Answer of exercise 3**

Yes, if \(E\) is open, then every \(x \in E\) is a limit point; this follows because every point is interior, i.e. \(N_\varepsilon(x) \subset E\), and \(N_\varepsilon(x)\) certainly contains points \(y \neq x\).

No, if \(E\) is closed, not every point \(x \in E\) need be a limit point; for example, if \(E = \{(0,0)\}\) say, then \(E\) has no limit points.

Exercise 4

Show that the union of a finite number of compact sets is compact.

**Answer of exercise 4**

Suppose \(K = \bigcup_{i=1}^{n} K_i\) with each \(K_i\) compact. Suppose given an open cover \(\{G_\alpha\}\) of \(K\). Since \(K_i\) is compact, there is a finite subset of \(\{G_\alpha\}\) which covers \(K_i\). Taking the union of these finite subsets gives a finite subset of \(\{G_\alpha\}\) which covers the whole \(K\). Thus every open cover has a finite subcover, as needed.

Exercise 5 \((\text{Rudin 2.14})\)

Show directly that the interval \((0,1) \subset \mathbb{R}\) is not compact, by giving an example of an open cover of \((0,1)\) which has no finite subcover.

**Answer of exercise 5**

Let \(G_n = \{x \mid 1/n < x < 1\} \subset \mathbb{R}\). The set \(\{G_n \mid n \in \mathbb{N}\}\) forms an open cover of \((0,1)\), since for any \(x \in (0,1)\) and any \(n > 1/x\), we have \(x \in G_n\). However, for any finite subcollection \(\{G_{n_k}\}\), letting \(N = \max\{n_1, \ldots, n_k\}\), any \(x < 1/N\) is not contained in any \(G_{n_k}\), and thus \(\{G_{n_k}\}\) cannot cover \((0,1)\).