Lecture 1

This is a course in <u>real analysis</u>.
So, we better start with the question, what is a <u>real number</u>?

Def A <u>field</u> is a set F equipped with binary operations $+$, $\times$ such that

[$\forall$ = "for all"/"for any"]

1) $x+y=y+x \quad \forall x,y\in F$
2) $(x+y)+z=x+(y+z) \quad \forall x,y,z\in F$
3) $\exists\, 0\in F$ s.t. $0+x=x \quad \forall x\in F$
4) $\forall x\in F\ \exists\,(-x)\in F$ s.t. $x+(-x)=0$
5) $xy=yx \quad \forall x,y\in F$
6) $(xy)z=x(yz) \quad \forall x,y,z\in F$
7) $\exists\, 1\in F$ s.t. $1\neq 0$, $1\cdot x=x \quad \forall x\in F$
8) $\forall x\in F$ s.t. $x\neq 0$, $\exists\,(1/x)\in F$ s.t. $x\cdot\left(\frac{1}{x}\right)=1$
9) $x(y+z)=xy+xz \quad \forall x,y,z\in F$

[$\exists$ = "there exists"]

Notation 1) Let $\mathbb{Z}$ denote the set of <u>integers</u>.
2) Let $\mathbb{Q}$ denote the set of <u>rational numbers</u>.
We assume their standard operations $(+,-,\times,\div)$ and relations $(>,<)$.
3) Let $\mathbb{N}=\{n\in\mathbb{Z} \mid n\geq 1\}$.

Ex $\mathbb{Q}$ is a field. (But not $\mathbb{Z}$.)

Prop If F is a field and $x,y,z\in F$ then
1) $x+y=x+z \Rightarrow y=z$
2) $x+y=x \Rightarrow y=0$ (zero is unique)
3) $x+y=0 \Rightarrow y=-x$
4) $-(-x)=x$

Pf of 1):
$$\begin{aligned} x+y=x+z &\Rightarrow -x+(x+y)=-x+(x+z)\\ &\Rightarrow (-x+x)+y=(-x+x)+z\\ &\Rightarrow 0+y=0+z\\ &\Rightarrow y=z \end{aligned}$$
$\blacksquare$

All the usual laws of arithmetic hold in any field F. From now on we'll use them freely.

Notation We write $x-y$ for $x+(-y)$ and x/y for $x \cdot \left(\frac{1}{y}\right)$.

Def An <u>ordered set</u> is a set S with a relation $<$ such that

1) $\forall x,y \in S$ exactly one of $x<y$, $y<x$ or $x=y$ is true
2) $\forall x,y,z \in S$ s.t. $x<y$ and $y<z$, $x<z$.

"such that"

Ex $\mathbb{Q}$ is an ordered set.

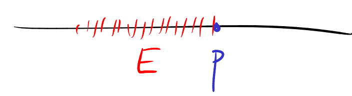
Notation We also write $x>y$ for $y<x$, and $x \geq y$ for ($x>y$ or $x=y$).

Def For S an ordered set and $E \subset S$,

1) <u>p is an upper bound for E</u> if $p \in S$ and $\forall q \in E$, $p \geq q$.

2) If $\exists p$ s.t. p is an upper bound for E, E is <u>bounded above</u>.

3) <u>p is a least upper bound for E</u> if



   a) p is an upper bound for E and
   b) if $q<p$ then q is not an upper bound for E

Similarly define <u>lower bound</u>, <u>greatest lower bound</u>.

Prop If $E \subset S$, p is a least upper bound for E, and q is a least upper bound for E, then $p=q$. ("Least upper bounds are unique")

Pf
p is LUB, q is UB $\implies p \leq q$ (by part 3b of def. of LUB)
q is LUB, p is UB $\implies q \leq p$. (also by part 3b of def. of LUB)
So $p=q$. $\blacksquare$

Notation If p is the LUB of E then we write $p = \sup E$.
" p " " GLB of E then we write $p = \inf E$.

Ex Take $S=\mathbb{Q}$ for all these examples. Then:

1) $E=\mathbb{Q}$: E has no least upper bound. (indeed no UB at all!)
2) $E=\{p \in \mathbb{Q} : 2 \leq p \leq 3\}$: $\sup E = 3$. (why?)
3) $E=\{p \in \mathbb{Q} : 2 \leq p < 3\}$: $\sup E = 3$. (why?)
4) $A=\{p \in \mathbb{Q} : p^2<2, p>0\}$: we don't know yet whether $\sup A$ exists...

"there does not exist"

Proposition $\nexists\, p \in \mathbb{Q}$ such that $p^2 = 2$

(There exists no $p \in \mathbb{Q}$ such that $p^2 = 2$.)

Pf Suppose $p \in \mathbb{Q}$, $p^2 = 2$.

Then $\exists\, m,n$ s.t. $p = \frac{m}{n}$ and m,n are not both even. (why?)

$$p^2 = 2 \Rightarrow \frac{m^2}{n^2} = 2, \text{ so } m^2 = 2n^2 \quad (*)$$

Thus m is even. (why?) But then $2n^2$ is divisible by 4, so n is even.

Thus both m and n are even $\divideontimes$. $\blacksquare$

"if and only if"

Prop p is a UB for A $\Longleftrightarrow$ $p^2 > 2$ and $p > 0$.

Pf ($\Leftarrow$) Suppose $p^2 > 2$, $p > 0$, and $q \in A$. Then $p^2 > 2 > q^2$, so $p^2 > q^2$.

If $q > p$, then $q^2 > qp > p^2$, so $q^2 > p^2$ $\divideontimes$

Thus, $\forall q \in A$, $p > q$. Thus p is a UB for A.

($\Rightarrow$) Suppose p is an UB for A.

If $p^2 < 2$, then consider $q = p - \frac{p^2 - 2}{p+2} > p$

$$q = \frac{2p+2}{p+2}$$

$$q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2} < 0, \text{ so } q \in A, \text{ but } q > p \quad \divideontimes$$

Thus $p^2 \geq 2$. But $p^2 \neq 2$, so $p^2 > 2$. Also $1 \in A$, so $p > 1 > 0$. $\blacksquare$

Prop A has no LUB in $\mathbb{Q}$.

Pf If p is an UB for A, then $p^2 > 2$ and $p > 0$.

Consider $q = p - \frac{p^2-2}{p+2} < p$. $q = \frac{2p+2}{p+2}$ so $q > 0$.

$$q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2} > 0 \text{ so } q^2 > 2$$

Thus q is an UB for A, but $q < p$!

So p cannot be LUB for A. $\blacksquare$

Def An <u>ordered field</u> is a field F which is also an ordered set, such that

1) $\forall x,y,z \in F$ s.t. $x+y<x+z$, $y<z$.
2) $\forall x,y \in F$ s.t. $x>0$ and $y>0$, $xy>0$.

Ex $\mathbb{Q}$ is an ordered field.

Prop If F is an ordered field, then $\forall x,y,z \in F$,

a) $x>0 \Rightarrow -x<0$
b) $x>0,\ y<z \Rightarrow xy<xz$
c) $x<0,\ y<z \Rightarrow xy>xz$
d) $x\neq 0 \Rightarrow x^2>0$
e) $0<x<y \Rightarrow 0<1/y<1/x$.

Pf a) $x>0 \Rightarrow -x+x>-x+0$, i.e. $0>-x$.

b) $y<z \Rightarrow z-y>0$. If also $x>0$, then $x(z-y)>0$. Thus $xy<xz$.

c,d,e): see Rudin.

Theorem $\exists$ an ordered field $\mathbb{R}$, such that

1) $\mathbb{Q}\subset\mathbb{R}$
2) $\forall E\subset\mathbb{R}$ bounded above, with $E\neq\phi$, E has a LUB in $\mathbb{R}$.

Pf ("Construction of the real numbers") See Rudin.

In particular, $A=\{p\in\mathbb{Q}\mid p^2<2,\ p>0\}$ has a LUB in $\mathbb{R}$, though it does <u>not</u> have one in $\mathbb{Q}$.

In this sense $\mathbb{R}$ "fills in the gaps" of $\mathbb{Q}$.

Moreover,

Prop $(\sup A)^2 = 2$.

Pf See Rudin.

Thm a) If $x, y \in \mathbb{R}$ and $x > 0$, $\exists n \in \mathbb{N}$ s.t. $nx > y$.

b) If $x, y \in \mathbb{R}$ and $x < y$, $\exists p \in \mathbb{Q}$ s.t. $x < p < y$.

Pf a) Suppose the contrary.

Let $A = \{nx \mid n \in \mathbb{Z}, n > 0\}$. y is an UB for A.

Thus A is a bounded subset of $\mathbb{R}$, hence $\sup A$ exists. Let $\alpha = \sup A$.

Then $\alpha - x$ is not an UB for A.

Thus $\exists m \in \mathbb{Z}$ s.t. $\alpha - x < mx$.

$$\alpha < (m+1)x \in A$$

But α is an UB for A! $\times\!\!\!\times$

b) $x < y \Rightarrow y - x > 0$.

Use a) to get: $\exists n \in \mathbb{N}$ s.t. $n(y-x) > 1$ $\quad (*)$

$\exists m_1 \in \mathbb{N}$ s.t. $m_1 > nx$

$\exists m_2 \in \mathbb{N}$ s.t. $m_2 > -nx$

Then $-m_2 < nx < m_1$, so $\exists m \in \mathbb{Z}$ s.t. $m - 1 \leq nx < m$,

ie $m \leq 1 + nx$ and $nx < m$.

Thus, $nx < m \leq 1 + nx < ny$

using $(*)$

and finally dividing by n (using $n > 0$),

$$x < \frac{m}{n} < y$$

as desired. ∎

Rk a) is sometimes called the "Archimedean property" of $\mathbb{R}$.
It says roughly that $\mathbb{R}$ does not contain infinitesimals.
There do exist ordered fields not satisfying this property...

Notation If $E \subset \mathbb{R}$ is not bounded above we write $\sup E = +\infty$.
If $E \subset \mathbb{R}$ is not bounded below we write $\inf E = -\infty$.