Stone-Weierstrass Theorem

Thm $f:[a,b]\to\mathbb{R}$ cts $\Rightarrow \exists$ a seq. of polynomials $\{P_n\}$ such that $P_n \to f$ uniformly on $[a,b]$

Pf Assume WLOG $[a,b]=[0,1]$ and $f(0)=f(1)=0$.

(can reduce to this case by taking $g(x)=f(x)-f(0)-x[f(1)-f(0)]$)

Extend f by $f(x)=0$ for $x\leq 0$, $x\geq 1$.

Then f is unif cts on $\mathbb{R}$.

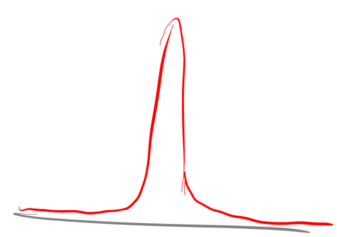
Set $Q_n(x)=c_n(1-x^2)^n$ with c_n chosen s.t. $\int_{-1}^{1} Q_n(x)\,dx=1$.

Note
$$\int_{-1}^{1}(1-x^2)^n\,dx = 2\int_0^1 (1-x^2)^n\,dx \geq 2\int_0^{1/\sqrt{n}}(1-x^2)^n\,dx$$
$\leftarrow$ compare derivatives on $(0,1)$
$$\geq 2\int_0^{1/\sqrt{n}}(1-nx^2)\,dx$$
$$= \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}},$$

so $c_n<\sqrt{n}$,

thus $Q_n(x)<\sqrt{n}(1-\delta^2)^n$ for $x\in[\delta,1]$

and thus $Q_n\to 0$ <u>uniformly</u> on $[\delta,1]$ for any δ



Set $P_n(x)=\int_{-1}^{1} f(x+t)\,Q_n(t)\,dt \qquad 0\leq x\leq 1$

$$\left[\begin{array}{ll} = \int_{-x}^{1-x} f(x+t)\,Q_n(t)\,dt & \text{since } f=0 \text{ outside } [0,1] \\ = \int_0^1 f(t)\,Q_n(t-x)\,dt & \underline{\text{evidently polynomial in } x} \end{array}\right]$$

Fix $\varepsilon>0$. Pick $\delta>0$ s.t. $|y-x|<\delta \Rightarrow |f(y)-f(x)|<\frac{\varepsilon}{2}$. Set $M=\sup\{|f(x)|\}$

$$|P_n(x)-f(x)| = \left|\int_{-1}^{1}(f(x+t)-f(x))\,Q_n(t)\,dt\right|$$
$$\leq \int_{-1}^{1}|f(x+t)-f(x)|\,Q_n(t)\,dt$$

$$\leq 2M\left[\int_{-1}^{-\delta} Q_n(t)\,dt + \int_{\delta}^{1} Q_n(t)\,dt\right] + \frac{\varepsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\,dt$$

$$< \quad \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

for large enough n. ∎

Cor For any a, $\exists$ a sequence of polynomials P_n s.t. $P_n(x) \to |x|$ uniformly on $[-a,a]$, with $P_n(0)=0$.

Def A collection $\mathcal{A}$ of functions $f: E \to \mathbb{R}$ is an algebra if; for all $f,g \in \mathcal{A}$

1) $f+g \in \mathcal{A}$
2) $fg \in \mathcal{A}$
3) $cf \in \mathcal{A}$ $\quad \forall c \in \mathbb{R}$

Def $\mathcal{A}$ is uniformly closed if $f_n \in \mathcal{A}$, $f_n \to f$ uniformly $\Longrightarrow f \in \mathcal{A}$.

$\mathcal{B} = \{ f \mid \exists \text{ seq. } \{f_n\} \text{ in } \mathcal{A} \text{ s.t. } f_n \to f \text{ uniformly}\}$ is the uniform closure of $\mathcal{A}$.

Thm If $\mathcal{A}$ is an algebra of bounded functions and $\mathcal{B}$ is uniform closure of $\mathcal{A}$ then $\mathcal{B}$ is a uniformly closed algebra.

Pf $f \in \mathcal{B}$, $g \in \mathcal{B}$ means $\exists \{f_n\}$ in $\mathcal{A}$, $f_n \to f$
$\{g_n\}$ in $\mathcal{A}$, $g_n \to g$

Then $f_n + g_n \to f+g$, $f_n g_n \to fg$, $cf_n \to cf$. So $\mathcal{B}$ is an algebra.

$\mathcal{B}$ is closure of $\mathcal{A}$ in the metric d introduced above, $d(f,g) = \|f-g\|$.
Thus $\mathcal{B}$ is closed in this metric, i.e. it's uniformly closed.

Def A collection $\mathcal{A}$ of functions $f: E \to \mathbb{R}$

1) separates points if $\forall x,y \in E$ with $x \neq y$, $\exists f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$.
2) vanishes nowhere if $\forall x \in E$, $\exists f \in \mathcal{A}$ s.t. $f(x) \neq 0$.

Prop If A sep pts on E and vanishes nowhere on E, and $x_1, x_2 \in E$, $x_1 \neq x_2$, $c_1, c_2 \in \mathbb{R}$ then $\exists f \in A$ s.t. $f(x_1) = c_1$, $f(x_2) = c_2$.

Pf easy (Rudin)

Thm (Stone-Weierstrass)
Let A be an algebra of cts functions on a compact set K.
So A sep pts, vanishes nowhere.
Then $\mathcal{B}$ = uniform closure of $A = \mathcal{C}(K)$.

Pf 1) If $f \in \mathcal{B}$ then $|f| \in \mathcal{B}$.

[Let $a = \sup\{|f(x)| : x \in K\}$, fix $\varepsilon > 0$.
$\exists c_1, \dots, c_n$ s.t. $\left|\sum_{i=1}^{n} c_i y^i - |y|\right| < \varepsilon \quad \forall y \in [-a, a]$
Then $g = \sum c_i f^i \in \mathcal{B}$, and $\big|g(x) - |f(x)|\big| < \varepsilon \quad \forall x \in K$.
And $\mathcal{B}$ is uniformly closed. Thus $|f| \in \mathcal{B}$.]

2) If $f \in \mathcal{B}$ and $g \in \mathcal{B}$ then $\max(f,g) \in \mathcal{B}$ and $\min(f,g) \in \mathcal{B}$.

[$\max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2}$, similarly $\min(f,g)$]

3) If $f : K \to \mathbb{R}$ cts, $x \in K$, $\varepsilon > 0$
then $\exists g_x \in \mathcal{B}$ s.t. $g_x(x) = f(x)$, $g_x(t) > f(t) - \varepsilon \quad \forall t \in K$.

[$\forall y \in K \ \exists h_y \in \mathcal{B}$ s.t. $h_y(x) = f(x)$, $h_y(y) = f(y)$.
h_y cts $\Rightarrow \exists$ nbhd J_y of y s.t. $h_y(t) > f(t) - \varepsilon \quad \forall t \in J_y$
K compact $\Rightarrow \exists y_1, \dots, y_n$ s.t. $K \subset J_{y_1} \cup \dots \cup J_{y_n}$
Put $g_x = \max(h_{y_1}, \dots, h_{y_n})$]

4) If $f : K \to \mathbb{R}$ cts, $x \in K$, $\varepsilon > 0$
then $\exists h \in \mathcal{B}$ s.t. $|h(x) - f(x)| < \varepsilon \quad \forall x \in K$

[g_x cts $\Rightarrow \exists$ nbhd V_x of x s.t. $g_x(t) < f(t) + \varepsilon \quad \forall t \in V_x$
K compact $\Rightarrow \exists x_1, \dots, x_m$ s.t. $K \subset V_{x_1} \cup \dots \cup V_{x_m}$

Put $h = \min(g_{x_1}, \dots, g_{x_m})$.

Cor If f is cts, $f(x+2\pi) = f(x)$ and $\varepsilon > 0$, then $\exists$ a trigonometric polynomial P such that $|P(x) - f(x)| < \varepsilon$ for all $x \in \mathbb{R}$.