Power series

Def If $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $x \in E$ to some $f(x)$ then we call $f(x)$ real analytic on E

Ex $f(x) = \frac{1}{1-x}$ is real analytic on $(-1,1)$

$f(x) = \sin(x)$ is real analytic on $\mathbb{R}$

For real analytic functions, we do have the standard version of Taylor's theorem. We'll prove it in a few steps.

Thm Say $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$ and define $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $|x| < R$

Then $\forall \varepsilon > 0$, $\sum c_n x^n$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$,

and $$f'(x) = \sum_{n=0}^{\infty} c_n \cdot n x^{n-1} \text{ for } |x| < R$$

Pf Fix $\varepsilon > 0$. For $|x| \leq R - \varepsilon$ we have

$$|c_n x^n| \leq |c_n (R-\varepsilon)^n|.$$

Since $\sum c_n (R-\varepsilon)^n$ converges absolutely [by root test] this shows that $\sum c_n x^n$ conv. uniformly on $[-R+\varepsilon, R-\varepsilon]$.

Since $\sqrt[n]{n} \to 1$ as $n \to \infty$ we have $\limsup_{n\to\infty} \sqrt[n]{n|c_n|} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$

thus $\sum c_n x^n$, $\sum c_n \cdot n x^{n-1}$ have same radius of convergence.

So $\sum c_n \cdot n x^{n-1}$ also conv. uniformly on $[-R+\varepsilon, R-\varepsilon]$ $\forall \varepsilon > 0$.

This gives $f'(x) = \sum_{n=0}^{\infty} c_n \cdot n x^{n-1}$ as desired, $\forall x \in [-R+\varepsilon, R-\varepsilon]$ and hence $\forall x \in (-R, R)$ since ε was arbitrary. ∎

Cor Say $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x|<R$

and define $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $|x|<R$

Then $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, c_n x^{n-k}$

In p'tic, $f^{(k)}(0) = k!\, c_k$.

So, if f comes to you already as a power series around $x=0$, then it is fully determined by its derivatives at $x=0$.

Lemma Suppose $\sum_{j=1}^{\infty} |a_{ij}| = b_i$, $\sum_{i=1}^{\infty} b_i$ converges. Then, $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$.

Pf We proved earlier that if a $\sum$ is absolutely convergent then every rearrangement is also convergent, to the same value. Could prove this the same way.

Instead, use a fun trick, leveraging what we proved about limit exchange for functions:

Let $E = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$

For each i, define a function $f_i : E \to \mathbb{R}$ by

$$f_i(0) = \sum_{j=1}^{\infty} a_{ij}, \quad f_i(\tfrac{1}{n}) = \sum_{j=1}^{n} a_{ij},$$

and $g : E \to \mathbb{R}$ by

$$g(x) = \sum_{i=1}^{\infty} f_i(x) \quad (x \in E).$$

Each f_i is cts at 0. Also $|f_i(x)| \le b_i$ $\forall x \in E$, so $\sum f_i(x)$ converges uniformly to $g(x)$. Thus, $g(x)$ is also cts at 0.

Then
$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} f_i(0) = g(0) = \lim_{n\to\infty} g(\tfrac{1}{n}) = \lim_{n\to\infty} \sum_{i=1}^{\infty} f_i(\tfrac{1}{n})$$
$$= \lim_{n\to\infty} \sum_{i=1}^{\infty}\sum_{j=1}^{n} a_{ij} = \lim_{n\to\infty} \sum_{j=1}^{n}\sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}. \qquad \blacksquare$$

(use $\sum c_n + d_n = \sum c_n + \sum d_n$)

Thm Suppose that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $|x|<R$, and $|a|<R$.

Then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ for $|x-a|<R-|a|$.

Pf

$$f(x) = \sum_{n=0}^{\infty} c_n\left((x-a)+a\right)^n$$

$$= \sum_{n=0}^{\infty} c_n \sum_{m=0}^{n} \binom{n}{m} a^{n-m}(x-a)^m$$

Now, since $\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left|c_n\binom{n}{m}a^{n-m}(x-a)^m\right| = \sum_{n=0}^{\infty}|c_n|\left(|x-a|+|a|\right)^n$ which converges when $|x-a|+|a|<R$, we may exchange order of summation using the previous Lemma to get

$$f(x) = \sum_{m=0}^{\infty}\left[\sum_{n=m}^{\infty}\binom{n}{m}c_n a^{n-m}\right](x-a)^m$$

Thus $f(x)$ is also a series around $x=a$.

Then the coefficients must be $\frac{f^{(n)}(a)}{n!}$ by the Corollary above. $\blacksquare$

Ex We defined

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

Now we know the same function can also be represented as a series expanded around any other point $x=a$.

(But, haven't proved it's periodic...)