**Def**  
1) Given two sets $A, B$, a function $f$ from $A$ to $B$ $(f : A \rightarrow B)$ is a rule which assigns an element $f(a) \in B$ for each $a \in A$.  
2) Given $f : A \rightarrow B$, $A$ is the **domain** of $f$  
$\{f(a) : a \in A\} \subseteq B$ is the **range** of $f$.  
3) If $E \subseteq A$, $f(E) = \{f(a) : a \in E\}$ is the **image** of $E$  
4) If $E \subseteq B$, $f^{-1}(E) = \{a : f(a) \in E\} \subseteq A$ is the **inverse image** of $E$  

**Ex**  
1) \[ \begin{align*} 
a_1 \cdot & \rightarrow \quad b_1 = f(a_1) = f(a_2) 
\quad a_2 \cdot & \rightarrow \quad b_2 
\quad a_3 \cdot & \rightarrow \quad b_3 
A \quad & \rightarrow \quad b_q = f(a_3) 
\end{align*} \]  
\[ \text{Range}(f) = f(A) = \{b_1, b_2\} \subseteq B \]  
f$\{a_1, a_2, b_3\}$  
f$^{-1}\{b_1, b_2\} = \{a_1, a_2\}$  

2) \[ f : \mathbb{Z} \rightarrow \mathbb{Z} \quad f(p) = p^2 \]  
f$\{1, 4, 9\}$  
f$^{-1}\{1, 4, 9\} = \{-1, -2, 2, -3, 3\}$  
f$^{-1}\{-1, -2, -3, \ldots\} = \emptyset$  

**Def**  
1) For $f : A \rightarrow B$, if $f(A) = B$ then $f$ is called **onto** or **surjective**.  
2) If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ then $f$ is called **1-1** or **injective**.  
3) If $f$ is both injective and surjective then $f$ is called **bijective**.  

**Ex**  
\[ f : \mathbb{Z} \rightarrow \mathbb{Z} \quad f(p) = 2p \]  
is **not surjective** since $f(\mathbb{Z}) \neq \mathbb{Z}$, e.g. $1 \notin f(\mathbb{Z})$  
is **injective** since $2p_1 = 2p_2 \Rightarrow p_1 = p_2$  
\[ f : \mathbb{Z} \rightarrow \mathbb{Z} \quad f(p) = p + 1 \]  
is **surjective** since $\forall q \in \mathbb{Z}$, $q = f(q-1)$  
is **injective** since $p + 1 = q + 1 \Leftrightarrow p = q$  
\[ f : A \rightarrow A \quad f(a) = a \]  
is **bijective**
Def  Given a set $E$:
1) $E$ is finite if $\exists n \in \mathbb{N}$ and $f: \{m \in \mathbb{N}, m \leq n\} \rightarrow E$ bijective.
2) $E$ is infinite if $E$ is not finite.
3) $E$ is countable if $\exists f: \mathbb{N} \rightarrow E$ bijective.
4) $E$ is uncountable if $E$ is not finite or countable.
5) $E$ is at most countable if $E$ is finite or countable.

Ex  1) $\mathbb{N}$ is countable. (f: $\mathbb{N} \rightarrow \mathbb{N}$, $f(n) = n$, is bijective)
2) $\mathbb{Z}$ is countable. Indeed $f: \mathbb{N} \rightarrow \mathbb{Z}$ $f(n) = \begin{cases} \frac{n}{2} & \text{n even} \\ -\frac{n-1}{2} & \text{n odd} \end{cases}$ is bijective.
   i.e. the list $-1, 1, -2, 2, -3, 3, ...$ contains each element of $\mathbb{Z}$ exactly once.

Prop  Every infinite subset of a countable set is countable.

Pf  ACB $f: \mathbb{N} \rightarrow B$ bijective. Define a sequence $\{m_n\}$ iteratively as follows: $m_1 =$ the smallest $\#$ s.t. $f(m_1) \in A$ $m_2 =$ the smallest $\# > m_1$ s.t. $f(m_2) \in A$ and $m_2 > m_1$ 
   $m_n =$ the smallest $\# > m_{n-1}$ s.t. $f(m_n) \in A$ and $m_n > m_{n-1}$ (why does such an $m_n$ exist?)
   Then define $g: \mathbb{N} \rightarrow A$ by $g(n) = f(m_n)$. (why is $g$ bijective?)
Next, unions and intersections.

**Def.** Suppose \( \Omega \) and \( A \) are sets, and for each \( \alpha \in A \) we have a set \( E_\alpha \subseteq \Omega \).

Then
\[
\bigcup_{\alpha \in A} E_\alpha = \{ x \in \Omega \mid \exists \alpha \in A \text{ s.t. } x \in E_\alpha \}
\]
\[
\bigcap_{\alpha \in A} E_\alpha = \{ x \in \Omega \mid \forall \alpha \in A, x \in E_\alpha \}
\]

**Notation.**

- If \( A = \{1, 2\} \) then \( \bigcup_{\alpha \in A} E_\alpha \) is also written as \( E_1 \cup E_2 \).
- If \( A = \mathbb{N} \) then \( \bigcup_{\alpha \in \Omega} E_\alpha \) is also written as \( \bigcup_{n=1}^{\infty} E_n \).
- Similarly for \( \bigcap \).

![Venn diagram](image_url)

\[
E_\infty = A_\infty = \{ x \in \mathbb{R} \mid x \geq n \}
\]

\[
A_{n_1} \cap A_{n_2} = A_{\max(n_1, n_2)} \quad A_{n_1} \cup A_{n_2} = A_{\min(n_1, n_2)}
\]

But, \( \bigcap_{n=1}^{\infty} A_n = \emptyset \)!
Thm If \( E_n \) is countable \( \forall n \in \mathbb{N} \) then \( \bigcup_{n=1}^{\infty} E_n \) is countable.

Pf Say \( E_n = \{x_{n1}, x_{n2}, x_{n3}, \ldots\} \)

\[
\begin{array}{cccc}
  & x_{11} & x_{12} & x_{13} & x_{14} \\
  x_{21} & x_{22} & x_{23} & x_{24} & \ldots \\
  x_{31} & x_{32} & x_{33} & x_{34} & \\
  x_{41} & x_{42} & x_{43} & x_{44} & \\
  \vdots & & & & \\
\end{array}
\]

then travel along diagonals: \( x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, x_{41}, x_{32}, x_{23}, x_{14}, \ldots \)

this list includes all elements of \( \bigcup_{n=1}^{\infty} E_n \)

(if some element is repeated, just skip it)

Thm If \( E \) is countable then the set of all \( n \)-tuples

\[
E^n = \{(x_1, \ldots, x_n) \mid x_i \in E \text{ for each } i\}
\]

is also countable.

Pf Induction on \( n \).

\( n = 1 \): \( E^1 = E \) countable \( \checkmark \)

So assume \( E^{n-1} \) countable. Then for each fixed \( \alpha = (x_1, \ldots, x_{n-1}) \in E^{n-1} \)

let \( E_{n, \alpha} = \{(x_1, \ldots, x_{n-1}, x_n) \mid x_n \in E\} \) be \( E^n = \bigcup_{\alpha \in E^{n-1}} E_{n, \alpha} \) which is countable union of countable sets.

\( \checkmark \)

Cn \( \mathbb{Q} \) is countable.

Pf \( \mathbb{Q} \subset \{(p, q) : p, q \in \mathbb{Z}\} \) so it's a subset of a countable set.
Def: If $E$ is a set, a sequence in $E$ is a function $f: \mathbb{N} \rightarrow E$.

(if $f(n) = x_n$, also write the sequence $x_1, x_2, x_3, ... = \{x_n\}$)

Thm: If $E$ has $>1$ element, the set of sequences in $E$ is uncountable.

Pf: Suppose $A \subseteq E$ countable, $A = \{s_1, s_2, s_3, ...\}$

- $s_1 = x_{11}, x_{12}, x_{13}, ...$
- $s_2 = x_{21}, x_{22}, x_{23}, ...$

Make a new sequence $\hat{s} = y_1, y_2, y_3, ...$

where $y_i \neq x_{ii}$

Then $\hat{s} \neq s_i \forall i \in \mathbb{N}$ (the two sequences are different in the $i$-th place)

Thus $\hat{s} \notin A$

Thus $A \neq E$. So $E$ cannot be countable.