Lecture 2

Def 1) Given two sets A, B a <u>function f from A to B</u> $(f: A \to B)$ is a rule which assigns an element $f(a) \in B$ for each $a \in A$.

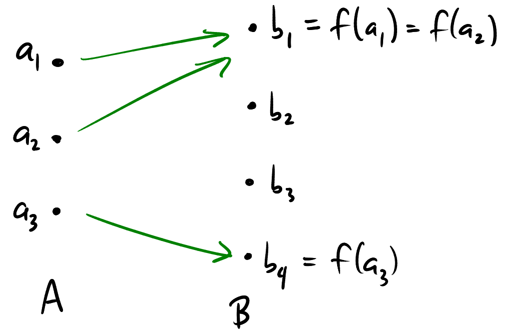
2) Given $f: A \to B$, A is the <u>domain</u> of f

$\{f(a) : a \in A\} \subset B$ is the <u>range</u> of f.

3) If $E \subset A$, $f(E) = \{f(a) \mid a \in E\}$ <u>image</u> of E

4) If $E \subset B$, $f^{-1}(E) = \{a \mid f(a) \in E\} \subset A$ <u>inverse image</u> of E

Ex 1)



$\text{Range}(f) = f(A) = \{b_1, b_4\} \subset B$

$f(\{a_1, a_2\}) = \{b_1\}$

$f^{-1}(\{b_1, b_3\}) = \{a_1, a_2\}$

2) $f: \mathbb{Z} \to \mathbb{Z}$ $\quad f(p) = p^2$

$f(\{1,2,3\}) = \{1,4,9\}$

$f^{-1}(\{1,4,9\}) = \{-1,1,-2,2,-3,3\}$

$f^{-1}(\{-1,-2,-3,\dots\}) = \emptyset$

Def 1) For $f: A \to B$, if $f(A) = B$ then f is called <u>onto</u> or <u>surjective</u>.

2) If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ then f is called <u>1-1</u> or <u>injective</u>.

3) If f is both injective and surjective then f is called <u>bijective</u>.

Ex $f: \mathbb{Z} \to \mathbb{Z}$ $\quad f(p) = 2p$ is <u>not surjective</u> since $f(\mathbb{Z}) \neq \mathbb{Z}$, e.g. $1 \notin f(\mathbb{Z})$

is <u>injective</u> since $2p_1 = 2p_2 \Rightarrow p_1 = p_2$

$f: \mathbb{Z} \to \mathbb{Z}$ $\quad f(p) = p+1$ is <u>surjective</u> since $\forall q \in \mathbb{Z}$, $q = f(q-1)$

<u>injective</u> since $p+1 = q+1 \Leftrightarrow p = q$

$f: A \to A$ $\quad f(a) = a$ is <u>bijective</u>

Def Given a set E:

1) E is finite if $\exists n \in \mathbb{N}$ and $f: \{m \in \mathbb{N},\ m \leq n\} \to E$ bijective.
2) E is infinite if E is not finite
3) E is countable if $\exists f: \mathbb{N} \to E$ bijective
4) E is uncountable if E is not finite or countable.
5) E is at most countable if E is finite or countable.

Ex 1) $\mathbb{N}$ is countable. ($f: \mathbb{N} \to \mathbb{N}$, $f(n) = n$, is bijective)

2) $\mathbb{Z}$ is countable. Indeed $f: \mathbb{N} \to \mathbb{Z}$ $\quad f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ -\frac{n-1}{2} & n \text{ odd} \end{cases}$

is bijective.

i.e. the list $-1, 1, -2, 2, -3, 3, \ldots$ contains each element of $\mathbb{Z}$ exactly once.

Prop Every infinite subset of a countable set is countable.

Pf $A \subset B$ $\quad f: \mathbb{N} \to B$ bijection. Define a sequence $\{m_n\}$ iteratively as follows:

$m_1 =$ the smallest # s.t. $f(m_1) \in A$

$m_2 =$ " " " " $f(m_2) \in A$ and $m_2 > m_1$

$\vdots$

$m_n =$ " " " " $f(m_n) \in A$ and $m_n > m_{n-1}$

(why does such an m_n exist?)

Then define $g: \mathbb{N} \to A$ by $g(n) = f(m_n)$.

(why is g bijective?)

∎

Next, unions and intersections.

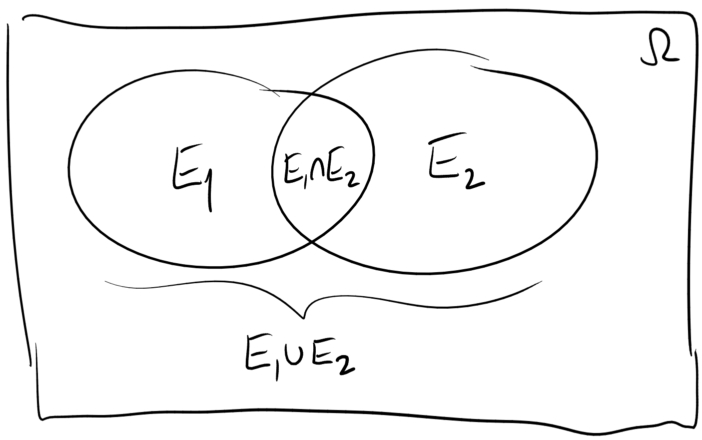
Def Suppose Ω and A are sets, and for each $\alpha \in A$ we have a set $E_\alpha \subset \Omega$.

Then
$$\bigcup_{\alpha \in A} E_\alpha = \{x \in \Omega \mid \exists \alpha \in A \text{ s.t. } x \in E_\alpha\}$$
$$\bigcap_{\alpha \in A} E_\alpha = \{x \in \Omega \mid \forall \alpha \in A,\ x \in E_\alpha\}$$

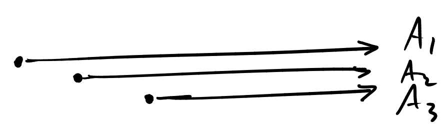
Notation If $A = \{1,2\}$ then $\bigcup_{\alpha \in A} E_\alpha$ is also written as $E_1 \cup E_2$

If $A = \mathbb{N}$ then $\bigcup_{\alpha \in \Omega} E_\alpha$ is also written as $\bigcup_{n=1}^{\infty} E_n$

similarly for $\cap$



Ex $A_n = \{x \in \mathbb{R} \mid x \geq n\}$



$$A_{n_1} \cap A_{n_2} = A_{\max(n_1,n_2)} \qquad A_{n_1} \cup A_{n_2} = A_{\min(n_1,n_2)}$$

But, $\bigcap_{n=1}^{\infty} A_n = \emptyset$!

Thm If E_n is countable $\forall n \in \mathbb{N}$ then $\bigcup_{n=1}^{\infty} E_n$ is countable

Pf Say $E_n = \{x_{n1}, x_{n2}, x_{n3}, \dots\}$

Make a grid:

$$\begin{matrix} x_{11} & x_{12} & x_{13} & x_{14} & \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \\ x_{41} & x_{42} & x_{43} & x_{44} & \\ & & \vdots & & \end{matrix}$$

then travel along diagonals: $x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, x_{41}, x_{32}, x_{23}, x_{14}, \dots$

this list includes all elements of $\bigcup_{n=1}^{\infty} E_n$

(if some element is repeated, just skip it) $\blacksquare$

Thm If E is countable then the set of all n-tuples $E^n = \{(x_1, \dots, x_n) \mid x_i \in E \text{ for each } i\}$ is also countable.

Pf Induction on n. $n=1$: $E^1 = E$ countable ✓

So assume E^{n-1} countable. Then for each fixed $\alpha = (x_1, \dots, x_{n-1}) \in E^{n-1}$

let $E_{n,\alpha} = \{(x_1, \dots, x_{n-1}, x_n) \mid x_n \in E\}$. $E^n = \bigcup_{\alpha \in E_{n-1}} E_{n,\alpha}$

which is countable union of countable sets. $\blacksquare$

Cor $\mathbb{Q}$ is countable.

Pf $\mathbb{Q} \subset \{(p,q) : p, q \in \mathbb{Z}\}$, so it's a subset of a countable set. $\blacksquare$

Def If E is a set, a sequence in E is a function $f: \mathbb{N} \to E$.
(if $f(n) = x_n$, also write the sequence $x_1, x_2, x_3, \ldots$ or $\{x_n\}$)

Thm If E has >1 element, the set of sequences in E is uncountable.

Pf Suppose $A \subset E$ countable, $A = \{s_1, s_2, s_3, \ldots\}$

$$s_1 = x_{11}, x_{12}, x_{13}, \ldots$$
$$s_2 = x_{21}, x_{22}, x_{23}, \ldots$$

Make a new sequence $\hat{s} = y_1, y_2, y_3, \ldots$
where $y_i \neq x_{ii}$

Then $\hat{s} \neq s_i \ \forall i \in \mathbb{N}$ (the two sequences are different in the i-th place)
Thus $\hat{s} \notin A$
Thus $A \neq E$! So E cannot be countable. ∎