# Lecture 3

**Def** For $x, y \in \mathbb{R}^n$, define $x \cdot y = x_1 y_1 + \cdots + x_n y_n$,
$|x| = (x \cdot x)^{1/2}$.

**Prop** a) $(x \cdot y)^2 \le |x|^2 |y|^2$ (Cauchy-Schwarz)
b) $|x+y| \le |x| + |y|$

**Pf** a) If $|y| = 0$ then $y = 0$ so $x \cdot y = 0$ ✓

Else, $$0 \le \Big|\underbrace{x - \frac{x\cdot y}{|y|^2} y}_{\perp \text{ projection of } x \text{ onto } y}\Big|^2 = |x|^2 - 2\frac{(x\cdot y)^2}{|y|^2} + \frac{(x \cdot y)^2}{|y|^2} = \frac{1}{|y|^2}\left(|x|^2|y|^2 - (x\cdot y)^2\right)$$

b) $|x+y|^2 = |x|^2 + 2x\cdot y + |y|^2$
$(|x| + |y|)^2 = |x|^2 + 2|x||y| + |y|^2$
Then use a). ∎

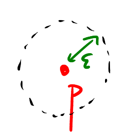
**Def** A <u>metric space</u> is a set $X$ plus a function $d : \{(p,q) : p, q \in X\} \to \mathbb{R}$ s.t. $\forall p, q, r \in X$,

a) $d(p,q) > 0$ if $p \ne q$,
b) $d(p,p) = 0$,
c) $d(p,q) = d(q,p)$
d) $d(p,q) \le d(p,r) + d(r,q)$

**Ex** 1) $\mathbb{R}^n$ is a metric space with the metric $d(x,y) = |x - y|$.
2) $\mathbb{R}^2$ is a metric space with "taxicab metric" $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$.

**Def** In a metric space $X$:

a) A <u>neighborhood</u> of $p \in X$ is a set $N_\varepsilon(p) = \{q \mid d(p,q) < \varepsilon\}$ where $\varepsilon > 0$.




b) $p$ is a <u>limit point</u> of $E \subset X$ if every neighborhood of $p$ contains some $q \in E$, $q \ne p$.

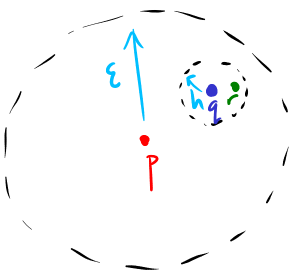
c) $E \subset X$ is <u>closed</u> if it contains all its limit points.

d) $p$ is an interior point of $E$ if $E$ contains a nbhd of $p$.

e) $E$ is open if all points of $E$ are interior.

**Thm** Every $N_\varepsilon(p)$ is open.

**Pf**




Say $q \in N_\varepsilon(p)$. Then let $h < \varepsilon - d(p,q)$.

For any $r \in N_h(q)$,
$$\begin{aligned} d(r,p) &< d(r,q) + d(q,p) \\ &< h + d(q,p) \\ &< \varepsilon - d(p,q) + d(q,p) \\ &= \varepsilon \end{aligned}$$

Thus $r \in N_\varepsilon(p)$.

So $N_h(q) \subset N_\varepsilon(p)$. Thus $q$ is interior pt of $N_\varepsilon(p)$. ■

**Ex** $(0,1) \subset \mathbb{R}$ is open.

**Thm** If $p$ is a limit pt of $E$ then every $N_\varepsilon(p)$ contains infinitely many points of $E$.

**Pf** Suppose $N_\varepsilon(p)$ contains only finitely many points $q_1, \dots, q_n \in E$.

Then let $h = \min\{d(p,q_i) \mid 1 \le i \le n\}$.

$N_{h/2}(p)$ does not contain any point of $E$.

But $p$ is a limit pt $\times\!\!\!\times$ ■

**Cor** If $E$ is finite then $E$ has no limit points.

**Ex** 1) Let $E = (a,b)$. Then any $x \in [a,b]$ is a limit point of $E$.

2) Let $E = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $x = 0$ is the only limit point of $E$.

3) Let $E = \mathbb{Q} \subset \mathbb{R}$. Then any $x \in \mathbb{R}$ is a limit point of $E$.

**Def** $E^c = \{x \in X \mid x \notin E\}$.

**Prop** $E$ is open $\iff$ $E^c$ is closed.

**Pf** $(\Rightarrow)$ Let $x$ be a limit pt of $E^c$. Then any nbhd of $x$ contains a point of $E^c$.

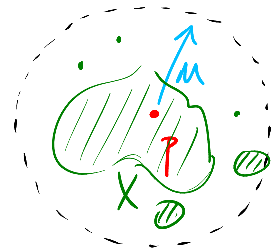
Thus $x$ cannot be an interior pt of $E$. But $E$ is open, so $x \notin E$, ie $x \in E^c$.

($\Leftarrow$) Let $x \in E$. Then $x \notin E^c$, so $x$ is not a limit pt of $E^c$. Thus $x$ has a nbhd $N$ which contains no point of $E^c$, i.e. $N \subset E$. Thus $x$ is an interior pt of $E$. ∎

**Cor** $E$ is closed $\Leftrightarrow$ $E^c$ is open.

**Rk** A set $E$ may be <u>neither</u> closed nor open, e.g. $E = (a,b] \subset \mathbb{R}$.
It may also be <u>both</u> closed and open, e.g. $E = \phi$ or $E = X$.

**Def** $E$ is <u>bounded</u> if $\exists p \in X$ and $M \in \mathbb{R}$ s.t. $\forall q \in X$, $d(p,q) < M$.




**Ex** $E \subset \mathbb{R}$: $E$ is bounded $\Leftrightarrow$ $E$ is bounded above and bounded below.

**Prop** $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$

**Pf** $\forall x \in X$:

$$x \in \left(\bigcup_\alpha E_\alpha\right)^c \Longleftrightarrow x \notin \bigcup_\alpha E_\alpha \Longleftrightarrow x \notin E_\alpha \ \forall \alpha \Longleftrightarrow x \in E_\alpha^c \ \forall \alpha \Longleftrightarrow x \in \bigcap_\alpha E_\alpha. \quad ∎$$

**Thm**
1) All $G_\alpha$ open $\Rightarrow \bigcup_\alpha G_\alpha$ open.
2) All $F_\alpha$ closed $\Rightarrow \bigcap_\alpha F_\alpha$ closed.
3) $G_1, \dots, G_n$ open $\Rightarrow \bigcap_{i=1}^n G_i$ open.
4) $F_1, \dots, F_n$ closed $\Rightarrow \bigcup_{i=1}^n F_i$ closed.

**Pf**
1) Say $x \in \bigcup_\alpha G_\alpha$. Then $\exists \alpha$ s.t. $x \in G_\alpha$. $G_\alpha$ open $\Rightarrow$ $\exists$ nbhd $N$ of $x$ with $N \subset G_\alpha \subset \bigcup_\alpha G_\alpha$. Thus $x$ is interior pt of $\bigcup_\alpha G_\alpha$.

2) $\forall \alpha$ $F_\alpha$ closed $\Rightarrow \forall \alpha$ $F_\alpha^c$ open $\Rightarrow \bigcup_\alpha F_\alpha^c$ open $\Rightarrow \left(\bigcap_\alpha F_\alpha\right)^c$ open $\Rightarrow \bigcap_\alpha F_\alpha$ closed.

3) Say $x \in \bigcap_{i=1}^n G_i$. Then $\forall i$, $\exists \varepsilon_i$ s.t. $N_{\varepsilon_i}(x) \subset G_i$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$. $N_\varepsilon(x) \subset G_i$ $\forall i$. So $N_\varepsilon(x) \subset \bigcap_{i=1}^n G_i$. Thus $x$ is an interior pt of $\bigcap_{i=1}^n G_i$.

4) similar to 2). ∎

**Rk** The finiteness is essential: $\bigcup_{n=1}^\infty G_n$ need not be closed even if all $G_n$ are closed!

(e.g. if $G_n = [\frac{1}{n}, 1]$ then $\bigcup_{n=1}^{\infty} G_n = (0,1]$ which has $0$ as a limit pt.)

**Def** If $E \subset X$, $E' = \{\text{limit pts of } E\}$, the <u>closure</u> of $E$ is $\overline{E} = E \cup E'$

**Prop** $\overline{E}$ is closed.

**Pf** Say $p \notin \overline{E}$. Then $p$ has a nbhd $N$ disjoint from $E$. If $N$ contains a point $q \in E'$ then it also contains some nbhd of $q$, which would contain a pt of $E$. So $N$ is also disjoint from $E'$! Thus $N \cap \overline{E} = \phi$. So $p$ is an interior pt of $\overline{E}^c$.
Thus $\overline{E}^c$ is open. ∎

**Cor** 1) $E = \overline{E} \iff E$ is closed.
2) If $E \subset F$ and $F$ is closed then $\overline{E} \subset F$.

**Pf** 1) exercise

2) $E' \subset F'$ follows easily from def of limit pt. $F$ closed so $F' \subset F$.
Thus $E' \subset F$ and $E \subset F$, so $E \cup E' \subset F$, i.e. $\overline{E} \subset F$. ∎

**Thm** $E \subset \mathbb{R}$ bounded above $\Longrightarrow \sup E \in \overline{E}$.

**Pf** Let $y = \sup E$. Suppose $\exists \varepsilon > 0$ s.t. $N_\varepsilon(y)$ disjoint from $E$. Then $y - \varepsilon$ is an upper bound for $E$. ※ ∎

NB, notion of "open" depends on the ambient metric space. e.g. any $E$ is open when considered as a subset of the metric space $E$.

**Thm** Say $E \subset Y \subset X$. Then $E$ is open in $Y \iff E = Y \cap G$ for $G \subset X$ open.

**Pf** ($\Rightarrow$) For each $p \in E$, $\exists \varepsilon_p > 0$ s.t. $\{q \in Y \mid d(p,q) < \varepsilon_p\} \subset E$.
Let $G = \bigcup_p \{q \in X \mid d(p,q) < \varepsilon_p\}$. Then check $E = Y \cap G$.

($\Leftarrow$) For $p \in E$, $\exists \varepsilon > 0$ s.t. $\{q \in X \mid d(p,q) < \varepsilon\} \subset G$.
Then $\{q \in Y \mid d(p,q) < \varepsilon\} \subset G \cap Y = E$. Thus $p$ is interior to $E$ in $Y$. ∎