Def For \( x, y \in \mathbb{R}^n \), define \( x \cdot y = x_1 y_1 + \cdots + x_n y_n \),
\[ |x| = (x \cdot x)^{1/2}. \]

Prop
\begin{enumerate}
  \item \((x \cdot y)^2 \leq |x|^2 |y|^2\) (Cauchy-Schwarz)
  \item \(|x + y|^2 \leq |x| + |y|^2\)
\end{enumerate}

Pf
\begin{enumerate}
  \item If \(|y| = 0\) then \(y = 0\) so \(x \cdot y = 0\).
  \item Else, \(0 \leq |x - \frac{x \cdot y}{|y|^2} y|^2 = |x|^2 - 2 \frac{(x \cdot y)^2}{|y|^2} + \frac{(x \cdot y)^2}{|y|^2} = \frac{1}{|y|^2} (|x|^2 |y|^2 - (x \cdot y)^2)\)
\end{enumerate}

Def A metric space is a set \( X \) plus a function \( d: \{(p, q) : p, q \in X\} \rightarrow \mathbb{R} \) s.t.
\( \forall p, q, r \in X, \)
\begin{enumerate}
  \item \(d(p, q) > 0\) if \(p \neq q\),
  \item \(d(p, p) = 0\),
  \item \(d(p, q) = d(q, p)\)
  \item \(d(p, q) \leq d(p, r) + d(r, q)\)
\end{enumerate}

Ex 1) \( \mathbb{R}^n \) is a metric space, with the metric \( d(x, y) = |x - y| \).
2) \( \mathbb{R}^2 \) is a metric space with "taxicab metric" \( d(x, y) = |x_1 - y_1| + |x_2 - y_2| \).

Def In a metric space \( X \):
\begin{enumerate}
  \item A neighborhood of \( p \in X \) is a set \( N_\varepsilon(p) = \{ q \mid d(p, q) < \varepsilon \} \) where \( \varepsilon > 0 \).
  \item \( p \) is a limit point of \( E \subseteq X \) if every neighborhood of \( p \) contains
    some \( q \in E, q \neq p \).
  \item \( E \subseteq X \) is closed if it contains all its limit points.
\end{enumerate}
d) $p$ is an interior point of $E$ if $E$ contains a nbhd of $p$.
e) $E$ is open if all points of $E$ are interior.

**Thm** Every $N_\varepsilon(p)$ is open.

**Pf**

Say $q \in N_\varepsilon(p)$. Then let $h = \varepsilon - d(p,q)$. For any $r \in N_h(q)$, $d(r,p) < d(r,q) + d(q,p)$

$$< h + d(q,p)$$

$$< \varepsilon - d(p,q) + d(q,p)$$

Thus $r \in N_\varepsilon(p)$.

So $N_h(q) \subset N_\varepsilon(p)$. Thus $q$ is interior pt of $N_\varepsilon(p)$.

**Ex** $(0,1) \subset \mathbb{R}$ is open.

**Thm** If $p$ is a limit pt of $E$ then every $N_\varepsilon(p)$ contains infinitely many points of $E$.

**Pf** Suppose $N_\varepsilon(p)$ contains only finitely many points $q_1, \ldots, q_n \in E$.

Then let $h = \min\{d(p,q_i) \mid 1 \leq i \leq n\}$.

$N_{h/2}(p)$ does not contain any point of $E$.

But $p$ is a limit pt $\times$

**Cor** If $E$ is finite then $E$ has no limit points.

**Ex**

1) Let $E = (a,b)$. Then any $x \in [a,b]$ is a limit point of $E$.

2) Let $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Then $x = 0$ is the only limit point of $E$.

3) Let $E = \mathbb{Q} \subset \mathbb{R}$. Then any $x \in \mathbb{R}$ is a limit point of $E$.

**Def** $E^c = \{x \in X \mid x \notin E\}$.

**Prop** $E$ is open $\iff$ $E^c$ is closed.

**Pf** ($\Rightarrow$) Let $x$ be a limit pt of $E^c$. Then any nbhd of $x$ contains a point of $E^c$.

Thus $x$ cannot be an interior pt of $E$. But $E$ is open, so $x \notin E$, i.e. $x \in E^c$. 
\[(\Leftarrow)\] Let \(x \in E\). Then \(x \notin E^c\), so \(x\) is not a limit pt of \(E^c\). Thus \(x\) has a nbhd \(N\) which contains no point of \(E^c\), i.e. \(N \subseteq E\). Thus \(x\) is an interior pt of \(E\).

**Cor**  \(E\) is closed \(\iff\) \(E^c\) is open.

**Rk** A set \(E\) may be neither closed nor open, e.g. \(E = (a, b] \subseteq \mathbb{R}\).

It may also be both closed and open, e.g. \(E = \emptyset\) or \(E = X\).

**Def** \(E\) is bounded if \(\exists p \in X\) and \(\exists M \in \mathbb{R}\) s.t. \(\forall q \in X, d(p, q) < M\).

**Ex** \(E \subseteq \mathbb{R}\): \(E\) is bounded \(\iff\) \(E\) is bounded above and bounded below.

**Pf** \((\bigcup_{\alpha} E_\alpha)^c = \bigcap_{\alpha} E_\alpha^c\)

**Pf** \(\forall x \in X:\)
\[x \in (\bigcup_{\alpha} E_\alpha)^c \iff x \notin \bigcup_{\alpha} E_\alpha \iff \forall \alpha, x \notin E_\alpha \forall \alpha \iff x \in \bigcap_{\alpha} E_\alpha \iff x \in \bigcap_{\alpha} E_\alpha^c\]

**Thm** 1) All \(G_\alpha\) open \(\Rightarrow\) \(\bigcup_{\alpha} G_\alpha\) open.
2) All \(F_\alpha\) closed \(\Rightarrow\) \(\bigcap_{\alpha} F_\alpha\) closed.
3) \(G_1, \ldots, G_n\) open \(\Rightarrow\) \(\bigcap_{i=1}^n G_i\) open.
4) \(F_1, \ldots, F_n\) closed \(\Rightarrow\) \(\bigcup_{i=1}^n F_i\) closed.

**Pf** 1) Say \(x \in \bigcup_{\alpha} G_\alpha\). Then \(\exists \alpha \text{ s.t. } x \in G_\alpha\). \(G_\alpha\) open \(\Rightarrow\) \(\exists\) nbhd \(N\) of \(x\) with \(N \subseteq G_\alpha\). Thus \(x\) is interior pt of \(\bigcup_{\alpha} G_\alpha\).
2) \(\forall \alpha F_\alpha\) closed \(\Rightarrow\) \(\forall \alpha F_\alpha^c\) open \(\Rightarrow\) \(\bigcup_{\alpha} F_\alpha^c\) open \(\Rightarrow\) \((\bigcup_{\alpha} F_\alpha^c)^c\) open \(\Rightarrow\) \(\bigcap_{\alpha} F_\alpha\) closed.
3) Say \(x \in \bigcap_{i=1}^n G_i\). Then \(\forall i, \exists \varepsilon_i\) s.t. \(N_{\varepsilon_i}(x) \subseteq G_i\). Let \(\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}\). \(N_{\varepsilon_i}(x) \subseteq G_i \forall i\). So \(N_{\varepsilon}(x) \subseteq \bigcap_{i=1}^n G_i\). Thus \(x\) is an interior pt of \(\bigcap_{i=1}^n G_i\).
4) Similar to 2).

**Rk** The finiteness is essential: \(\bigcup_{n=1}^\infty G_n\) need not be closed even if all \(G_n\) are closed!
(e.g. if $G_n = \left( \frac{1}{n}, 1 \right]$ then $\bigcup_{n=1}^{\infty} G_n = (0, 1]$ which has 0 as a limit pt.)

**Def.** If $E \subset X$, $E' = \{\text{limit pts of } E\}$, the closure of $E$ is $\overline{E} = E \cup E'$.

**Prop.** $\overline{E}$ is closed.

**Pf.** Say $p \in \overline{E}$. Then $p$ has a nhbd $N$ disjoint from $E$. If $N$ contains a pt $q \in E'$ then it also contains some nhbd of $q$, which would contain a pt of $E$. So $N$ is also disjoint from $E'$. Thus $N \cap E' = \emptyset$. So $p$ is an interior pt of $E'$.

Thus $\overline{E}$ is open.

**Cor.**

1) $E = \overline{E} \iff E$ is closed.

2) If $E \subset F$ and $F$ is closed then $\overline{E} \subset F$.

**Pf.**

1) exercise

2) $E' \subset F'$ follows easily from def of limit pt. $F$ closed so $F'$ is $F$.

Thus $E' \subset F'$ and $E \subset F$ so $\overline{E} \subset F$, i.e. $E \subset F$.

**Thm.** $E \subset X$. Then $E$ is open in $Y \iff E = Y \cap G$ for $G \subset X$ open.

**Pf.** ($\Rightarrow$) For each $p \in E$, $\exists \varepsilon > 0$ s.t. $\{q \in Y \mid d(p, q) < \varepsilon\} \subset E$.

Let $G = \bigcup \{q \in X \mid d(p, q) < \varepsilon\}$. Then check $E = Y \cap G$.

($\Leftarrow$) For $p \in E$, $\exists \varepsilon > 0$ s.t. $\{q \in X \mid d(p, q) < \varepsilon\} \subset G$.

Then $\{q \in Y \mid d(p, q) < \varepsilon\} \subset G \cap Y = E$. Thus $p$ is interior to $E$ in $Y$. 

NB, notion of "open" depends on the ambient metric space. e.g. any $E$ is open when considered as a subset of the metric space $E$.