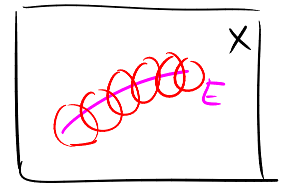
Lecture 4

X metric space, $E \subset X$.



Def An open cover of E is a collection of open sets G_α s.t. $E \subset \bigcup_\alpha G_\alpha$.

Ex $E = [0,1]$ $\quad G_1 = (-2, \frac{1}{2})$ $\quad G_2 = (\frac{1}{4}, 3)$ $\quad G_3 = (\frac{3}{4}, 2)$

Ex $E = (0, \infty)$ $\quad G_n = (n-1, n+\frac{1}{10})$ $\quad n \in \mathbb{N}$

Def A finite subcover of an open cover $\{G_\alpha\}$ is a set $\{G_{\alpha_1}, \dots, G_{\alpha_n}\} \subset \{G_\alpha\}$ s.t. $E \subset \bigcup_{i=1}^{n} G_{\alpha_i}$.

Ex $E = (0,1)$ $\quad G_x = (x - \frac{1}{10}, x + \frac{1}{10})$ $\quad \{G_x \mid x \in \mathbb{R}\}$ is a cover

$\{G_0, G_{1/10}, G_{2/10}, \dots, G_1\}$ is a finite subcover

$E = (0,1]$ $\quad G_n = (\frac{1}{n}, 2)$ $\quad \{G_n \mid n \in \mathbb{N}\}$ is a cover which has no finite subcover!

Def K is compact if every open cover of K has a finite subcover.

Ex $(0,1]$ is not compact.

Prop Any finite set is compact.

Morally, we should view "compact" as a class of sets which have many of the nice properties that finite sets have.

The def^n of compact in principle depends on which metric space E is a subset of. But in practice this doesn't matter:

Prop If $K \subset X \subset Y$, then K is compact in $X \iff K$ is compact in Y.

Pf ($\Rightarrow$) Suppose $\{G_\alpha\}$ is an open cover of K in Y.

Then let $H_\alpha = G_\alpha \cap X$. $\{H_\alpha\}$ is an open cover of K in X.

Thus $\{H_\alpha\}$ has a finite subcover $\{H_{\alpha_1}, \dots, H_{\alpha_n}\}$.

$\{G_{\alpha_1}, \dots, G_{\alpha_n}\}$ is a finite subcover of $\{G_\alpha\}$.

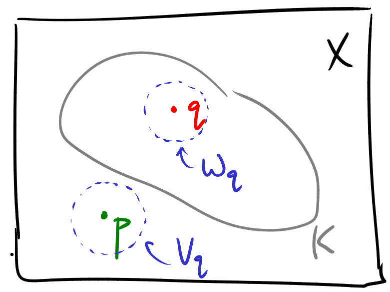
($\Leftarrow$) Suppose $\{H_\alpha\}$ is an open cover of K in X.

Then $H_\alpha = G_\alpha \cap X$ for some G_α open in Y. $\{G_\alpha\}$ is an open cover of

K in Y. Thus $\{G_\alpha\}$ has finite subcover $\{G_{\alpha_1}, \ldots, G_{\alpha_n}\}$.
$\{H_{\alpha_1}, \ldots, H_{\alpha_n}\}$ is a finite subcover of $\{H_\alpha\}$. ∎

Thm K compact $\Rightarrow$ K closed.

Pf We'll prove K^c is open.
Say $p \in K^c$ and $q \in K$.
Let V_q and W_q be nbhds of p and q with radii $< \frac{1}{2} d(p,q)$.



$\{W_q \mid q \in K\}$ is an open cover of $K \Rightarrow$ has finite subcover, $\{W_{q_1}, \ldots, W_{q_n}\}$.
$V_{q_1} \cap \cdots \cap V_{q_n}$ is a nbhd of p which is disjoint from K. ∎

Thm K compact, $F \subset K$ closed $\Rightarrow$ F compact.

Pf Say $\{V_\alpha\}$ open cover of F. Then $\{V_\alpha\} \cup \{F^c\}$ is open cover of K.
Thus it has a finite subcover. Removing F^c if necessary gives a finite subcover of $\{V_\alpha\}$. ∎

Cor F closed, K compact $\Rightarrow$ $F \cap K$ compact.

Thm If $\{K_\alpha\}$ is a collection of compact sets s.t. finite intersections $K_{\alpha_1} \cap \cdots \cap K_{\alpha_n}$ are all nonempty, then $\bigcap_\alpha K_\alpha \neq \emptyset$.

Pf Suppose $\bigcap_\alpha K_\alpha = \emptyset$. Set $G_\alpha = K_\alpha^c$. The G_α form an open cover of K_1.
It has a finite subcover. But this means some $K_{\alpha_1} \cap \cdots \cap K_{\alpha_n}$ is empty! ※ ∎

Cor If $K_1 \supset K_2 \supset K_3 \supset \cdots$ with all K_n compact then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

Thm If $E \subset K$, K compact, E infinite then E has a limit point in K.

Pf Assume not. Then $\forall q \in K$, $\exists$ nbhd V_q of q s.t. $V_q \cap K \subset \{q\}$.
This is an open cover of K w/ no finite subcover. ※ ∎

Now we want to prove that $[a,b] \subset \mathbb{R}$ is compact. Begin with:

Lemma If $\{I_n\}$ is a sequence of closed intervals in $\mathbb{R}$, such that $I_1 \supset I_2 \supset I_3 \supset \cdots$ then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Pf Say $I_n = [a_n, b_n]$. Let $E = \{a_n \mid n \in \mathbb{N}\}$. All $a_n \leq b_1$ so E is bounded above.
Let $x = \sup E$. Thus $\forall m \in \mathbb{N}$ $a_m \leq x$.
$\forall m,n \in \mathbb{N}$, $a_n \leq a_{m+n} \leq b_{m+n} \leq b_m$
thus b_m is an upper bound for E, so $x \leq b_m$. Thus $x \in [a_m, b_m]$.
Thus $x \in \bigcap_{m=1}^{\infty} I_m$ ∎

Theorem Any closed interval $I \subset \mathbb{R}$ is compact.

Pf Say $I = [a,b]$.
Suppose $\exists$ an open cover $\{G_\alpha\}$ of I w/ no finite subcover.
Say $c = \frac{1}{2}(a+b)$. Then $\{G_\alpha\}$ also is open cover of both $[a,c]$ and $[c,b]$. At least one of these two has no finite subcover.
Bisect this interval again and continue. We get an infinite chain $I_1 \supset I_2 \supset \cdots$ such that no I_n can be covered by any finite subcover of $\{G_\alpha\}$, and any $x,y \in I_n$ have $|x-y| < \frac{a+b}{2^n}$.
Now, by the Lemma, $\exists x \in \bigcap_{n=1}^{\infty} I_n$. Since $\{G_\alpha\}$ covers I, $\exists \alpha$ s.t. $x \in G_\alpha$. Since G_α is open, $\exists \varepsilon > 0$ s.t. $N_\varepsilon(x) \subset G_\alpha$.
But $\exists n$ s.t. $\frac{a+b}{2^n} < \varepsilon$ (why?) and hence $I_n \subset G_\alpha$. ⨳

Rk The same is true for a closed k-cell in $\mathbb{R}^k$,

$$\{(x_1, \ldots, x_k) \mid a_1 \leq x_1 \leq b_1,\ a_2 \leq x_2 \leq b_2, \ldots, a_k \leq x_k \leq b_k\}.$$

(See Rudin — similar to above.)

Now we can understand all the compact subsets of $\mathbb{R}$:

Theorem For $E \subset \mathbb{R}$, the following are equivalent:

a) E is closed and bounded

b) E is compact

c) every infinite subset of E has a limit point in E

Pf a) $\Rightarrow$ b): Say E closed and bounded.
E bounded $\Rightarrow E \subset I$ for some closed interval I.
So E is a closed subset of a compact set. Thus E is compact.

b) $\Rightarrow$ c): Already proved.

c) $\Rightarrow$ a): Say every infinite subset of E has a limit point in E.
Suppose E not bounded.
Then $\forall n \in \mathbb{N}\ \exists x_n \in E$ s.t. $|x_n| > n$.
$\{x_n\}$ has no limit point (why?) ※
So E is bounded.

Suppose E not closed.
Then $\exists x_0 \in \mathbb{R}$ s.t. $x_0 \notin E$, x_0 a limit point of E.
$\forall n \in \mathbb{N}$, $\exists x_n \in E$ s.t. $|x_n - x_0| < \frac{1}{n}$. Let $S = \{x_n\}$.
For $y \in \mathbb{R}$,
$$|x_n - y| \geq |x_0 - y| - |x_n - x_0| \geq |x_0 - y| - 1/n$$
Suppose $y \neq x_0$.
Then for $n \geq \frac{2}{|x_0 - y|}$ we get $|x_n - y| \geq \frac{1}{2|x_0 - y|}$

So y is not a limit point of S.

Thus S has no limit point in E. ※ ∎

Rk The above thm is also true if we replace $\mathbb{R}$ by $\mathbb{R}^k$.
See Rudin for the proof. (Similar to what we did for $\mathbb{R}$.)
But, it's <u>not</u> true for a general metric space!

Thm $E \subset \mathbb{R}^k$ bounded, infinite $\Rightarrow$ E has a limit point in $\mathbb{R}^k$.

Pf E is contained in some closed k-cell I.
I is compact. Thus E has a limit point in I.

Def X metric space: $E \subset X$ is <u>perfect</u> if E is closed and every point of E is a limit point.

Ex $E = [a,b] \subset \mathbb{R}$ is perfect.

Thm $E \subset \mathbb{R}^k$ is perfect, $E \neq \emptyset$ $\Rightarrow$ E is uncountable.

Pf E has limit pts $\Rightarrow$ E is infinite. Suppose E is countable. $E = \{x_1, x_2, \dots\}$
Let V_1 be a nbhd of x_1. We'll inductively construct a sequence $\{V_1, V_2, \dots\}$ of nbhds such that:

- $V_n \cap E \neq \emptyset$
- $x_{n-1} \notin \overline{V_n}$ (for $n>1$)
- $\overline{V_n} \subset V_{n-1}$ (for $n>1$)

(Namely, $V_n = N_{\varepsilon_n}(y_n)$ where $y_n \in V_{n-1} \cap E$, $y_n \neq x_{n-1}$, $\varepsilon_n < |y_n - x_{n-1}|$ and $\varepsilon_n < \varepsilon_{n-1} - |y_n - y_{n-1}|$; to see that such a y_n exists, use the fact that x_{n-1} is a limit point)

Let $K_n = \overline{V_n} \cap E$. $\overline{V_n}$ is closed, bounded, thus compact; and E closed; thus K_n also compact. Also $K_n \supset K_{n+1} \supset \cdots$ so $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. But $\bigcap_{n=1}^{\infty} K_n = \emptyset$ (since $x_{n-1} \notin K_n$ and $K_n \subset E$). ※ ∎

Cor Any closed interval in $\mathbb{R}$ is uncountable.

Cor $\mathbb{R}$ is uncountable.