Lecture 6

X metric space

Def A sequence $\{p_n\}$ in X <u>converges to</u> $p \in X$ if $\forall \varepsilon > 0 \; \exists n \in \mathbb{N}$ s.t. $n \geq N \Rightarrow d(p, p_n) < \varepsilon$.

Notation If $\{p_n\}$ converges to p we write $p_n \to p$ or $\lim_{n\to\infty} p_n = p$.

Def If $\exists p$ s.t. $p_n \to p$, we call $\{p_n\}$ <u>convergent</u>.
Otherwise we call $\{p_n\}$ <u>divergent</u>.

Ex $p_n = 1/n \qquad p_n \to 0$, [because $\forall \varepsilon > 0$, and any $N \geq \frac{1}{\varepsilon}$, $n \geq N \Rightarrow |p_n| < \varepsilon$]

Ex $p_n = 1 \qquad p_n \to 1$ [because $\forall \varepsilon > 0$, and <u>any</u> $N \in \mathbb{N}$, $n \geq N \Rightarrow |p_n - 1| < \varepsilon$]

Ex $p_n = n \qquad p_n$ diverges

Ex $p_n = (-1)^n \qquad p_n$ diverges

Thm $\{p_n\}$ a sequence in X:

1) $p_n \to p \iff$ any nbhd of p contains all but finitely many of the p_n
2) $p_n \to p$ and $p_n \to p' \Rightarrow p = p'$
3) $\{p_n\}$ converges $\Rightarrow \{p_n\}$ bounded.
4) $E \subset X$, p limit pt of $E \Rightarrow \exists$ a sequence $\{p_n\}$ in E with $p_n \to p$.

Pf 1) $(\Rightarrow)$ Take any nbhd $N_\varepsilon(p)$. $\exists N$ s.t. $\forall n \geq N$, $p_n \in N_\varepsilon(p)$.
$(\Leftarrow)$ Take $\varepsilon > 0$. $N_\varepsilon(p)$ contains all but finitely many p_n, so $\exists N$ s.t. $\forall n \geq N$, $p_n \in V$, ie $p_n \in N_\varepsilon(p)$.

2) Fix $\varepsilon = d(p, p')$. $\exists N$ s.t. $n \geq N \Rightarrow d(p_n, p) < \varepsilon/2$
$\exists N'$ s.t. $n \geq N' \Rightarrow d(p_n, p') < \varepsilon/2$

Thus for $n \geq \max(N, N')$ $d(p, p') < d(p, p_n) + d(p_n, p') < \varepsilon$. ※

3) Say $p_n \to p$. $\exists N$ s.t. $n \geq N \Rightarrow d(p,p_n) < 1$.
Set $M = \max(1, d(p,p_1), d(p,p_2), \dots, d(p,p_n))$.
$\forall n \in \mathbb{N}$ $d(p,p_n) < M$.

4) $\forall n \in \mathbb{N}$, $\exists p_n \in E$ s.t. $d(p,p_n) < \frac{1}{n}$. Build the sequence $\{p_n\}$.
$\forall \varepsilon > 0$, pick $N > \frac{1}{\varepsilon}$. Then if $n \geq N$, $d(p,p_n) < \varepsilon$. So $p_n \to p$ ∎

<u>Thm</u> Suppose $\{s_n\}$, $\{t_n\}$ sequences with $s_n \to s$, $t_n \to t$. Then

a) $(s_n + t_n) \to s + t$

b) $(c \cdot s_n) \to c \cdot s$, $(c + s_n) \to c + s$

c) $s_n \cdot t_n \to s \cdot t$

d) $\frac{1}{s_n} \to \frac{1}{s}$ (assuming all $s_n \neq 0$ and $s \neq 0$)

<u>Pf</u> a) Fix $\varepsilon > 0$. $\exists N_1 \in \mathbb{N}$ s.t. $n \geq N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$.
$\exists N_2 \in \mathbb{N}$ s.t. $n \geq N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{2}$.
Let $N = \max(N_1, N_2)$. Then for $n \geq N$, $|(s_n + t_n) - (s+t)| < |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$.

b) Exercise

c) Rudin.

d) Fix $\varepsilon > 0$. $\exists N_1$ s.t. $|s_n - s| < \frac{1}{2}|s|$ for $n \geq N_1$. Then $|s_n| \geq |s| - |s_n - s| > \frac{1}{2}|s|$ for $n \geq N_1$.
$\exists N_2$ s.t. $|s_n - s| < \frac{1}{2}|s|^2 \varepsilon$ for $n \geq N_2$. Let $N = \max(N_1, N_2)$.

For $n \geq N$, $\left| \frac{1}{s_n} - \frac{1}{s} \right| = \frac{|s_n - s|}{|s_n||s|} < \frac{\frac{1}{2}|s|^2 \varepsilon}{\frac{1}{2}|s|^2} = \varepsilon$. ∎

<u>Def</u> Given a sequence $\{p_n\}$, consider any positive integers $n_1 < n_2 < n_3 < \cdots$
The sequence $\{p_{n_k}\}$ is called a <u>subsequence</u> of $\{p_n\}$.

<u>Thm</u> If $\{p_n\}$ is a sequence in a compact set K, then some subsequence of $\{p_n\}$ converges.

<u>Pf</u> Let E be the range of $\{p_n\}$, i.e. $E = \{p_n \mid n \in \mathbb{N}\}$. If E is finite, then there is a <u>constant</u> subsequence. If E is infinite, then it has a limit point p, and hence $\exists$ a sequence in E which converges to p. ∎

Cor If $\{p_n\}$ is a bounded sequence in $\mathbb{R}^n$, then some subsequence of $\{p_n\}$ converges.

Def A sequence $\{p_n\}$ in a metric space X is Cauchy if $\forall \varepsilon > 0\ \exists N$ s.t.

$$n, m \geq N \Rightarrow |p_n - p_m| < \varepsilon.$$

Def $\operatorname{diam}(E) = \sup \{ d(p,q) \mid p, q \in E \}$.

Thm a) $\operatorname{diam}(\overline{E}) = \operatorname{diam}(E)$.

b) If $K_n \supset K_{n+1} \supset \cdots$ and each K_n is compact, and $K_n \to 0$,

then $\bigcap_{n=1}^{\infty} K_n$ contains exactly 1 point.

Pf See Rudin.

Thm a) In any metric space X, all convergent sequences are Cauchy.

b) X compact, $\{p_n\}$ Cauchy seq in $X \Rightarrow \exists p \in X$ s.t. $p_n \to p$.

c) $\{p_n\}$ Cauchy seq in $\mathbb{R}^k \Rightarrow \exists p \in \mathbb{R}^k$ s.t. $p_n \to p$.

Pf a) If $p_n \to p$, $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |p - p_n| < \frac{\varepsilon}{2}$.

Thus if $n, m \geq N$, $|p_n - p_m| \leq |p_n - n| + |p_m - n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

b) $\forall N \in \mathbb{N}$, let $E_N = \{p_n \mid n \geq N\}$.

Then $\lim_{N \to \infty} \operatorname{diam} \overline{E}_N = 0$, and each $\overline{E}_N$ is compact, with $\overline{E}_N \supset \overline{E}_{N+1} \supset \cdots$

So $\exists$ a unique $p \in X$ s.t. $p \in \overline{E}_N\ \forall N \in \mathbb{N}$. (We don't actually need the uniqueness here)

$\forall \varepsilon > 0$, $\exists N_0$ s.t. $\operatorname{diam} \overline{E}_N < \varepsilon$ for $N \geq N_0$.

Then for $n \geq N_0$, $p_n \in \overline{E}_N$ and $p \in \overline{E}_N$, so $d(p_n, p) < \varepsilon$. Thus $p_n \to p$.

c) Cauchy seq are bounded (why?) so they are contained in some cells.

Then use b).

Def If X is a metric space in which every Cauchy sequence converges, then call X complete.

Cor 1) $\mathbb{R}^k$ is complete.

2) Any compact metric space is complete.

Rk $\mathbb{Q}$ is not complete. (why?)

Def A sequence $\{s_n\}$ in $\mathbb{R}$ is <u>monotonically increasing</u> if $\forall n$ $s_{n+1} \geq s_n$

" " " " " " " <u>decreasing</u> if $\forall n$ $s_{n+1} \leq s_n$.

<u>monotonic</u> if either mon. incr. or mon. decr.

Thm If $\{s_n\}$ monotonic, then $\{s_n\}$ converges $\iff$ $\{s_n\}$ bounded.

Pf $(\Longrightarrow)$ already proved.

$(\Longleftarrow)$ Say $\{s_n\}$ mon. increasing.

Let $s = \sup \{s_n \mid n \in \mathbb{N}\}$.

$\forall \varepsilon > 0, \exists N$ s.t. $s_N > s - \varepsilon$. But then $\forall n \geq N$, $|s_n - s| < \varepsilon$. ∎

If $\{s_n\}$ is a sequence in $\mathbb{R}$: $\lim_{n\to\infty} s_n$ need not exist, but here's something that does.
Much more robust notion, e.g. survives under small perturbations of all the s_n.

Def If $\forall M \in \mathbb{R}$ $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow s_n > M$, we write $s_n \to \infty$.

Def If $\forall M \in \mathbb{R}$ $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow s_n < M$, we write $s_n \to -\infty$.

Def $\limsup_{n\to\infty} s_n = \sup \{p \in \underbrace{\mathbb{R} \cup \{\infty, -\infty\}}_{\text{"extended real numbers"}} \mid \exists \text{ a subsequence } p_{n_k} \to p\} \in \mathbb{R} \cup \{\infty, -\infty\}$

Similarly define $\liminf_{n\to\infty} s_n$.

Ex $s_n = (-1)^n$: $\limsup_{n\to\infty} s_n = 1$

$\liminf_{n\to\infty} s_n = -1$

Ex $s_n = n$: $\limsup_{n\to\infty} s_n = \infty$

$\liminf_{n\to\infty} s_n = \infty$

Ex s_n is a sequence containing all rationals: $\limsup_{n\to\infty} s_n = \infty$

$\liminf_{n\to\infty} s_n = -\infty$

(every $x \in \mathbb{R}$ is limit of some subsequence!)

Prop 1) $\liminf_{n\to\infty} s_n \le \limsup_{n\to\infty} s_n$.

2) $s_n \to s \iff \limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$.

3) If $\exists N \in \mathbb{N}$ s.t. $\forall n \ge N$, $s_n \le t_n$, then $\liminf s_n \le \liminf t_n$
$\limsup s_n \le \limsup t_n$

4) If $\alpha > \limsup_{n\to\infty} s_n$ then $\exists N$ s.t. $\forall n \ge N$, $s_n < \alpha$.

Pf Exercise.

Cor If $\exists N$ s.t. $0 \le x_n \le s_n$ for $n \ge N$, and $s_n \to 0$, then $x_n \to 0$.

Pf $0 \le \limsup_{n\to\infty} x_n \le 0$ and $0 \le \liminf_{n\to\infty} x_n \le 0$, so $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = 0$. ∎

Prop 1) If $p > 0$, $\frac{1}{n^p} \to 0$.

2) If $p > 0$, $p^{1/n} \to 1$.

3) $\sqrt[n]{n} \to 1$.

4) If $p > 0$ and $\alpha \in \mathbb{R}$, $\frac{n^\alpha}{(1+p)^n} \to 0$.

5) If $|x| < 1$, $x^n \to 0$.

Pf 1) Take $N > (1/\varepsilon)^{1/p}$. Then $\forall n \ge N$, $\left|\frac{1}{n^p}\right| < \varepsilon$.

2) For $p > 1$, let $x_n = \sqrt[n]{p} - 1$. $x_n > 0$, we want to show that $x_n \to 0$.

Know $(1+x_n)^n = p$; but $1 + nx_n \le (1+x_n)^n$ by binomial thm.

Thus $1 + nx_n \le p$

$x_n \le \frac{p-1}{n}$

and already know $0 \le x_n$, so $0 \le x_n \le \frac{p-1}{n}$, and $\frac{p-1}{n} \to 0$, so $x_n \to 0$, so $\sqrt[n]{p} \to 1 + 0 = 1$.

3,4,5) more elaborate uses of same basic idea: see Rudin. ∎