Limits and Continuity

Def Suppose $E \subset X$, p is a limit point of E, and $f: E \to Y$.
Then, we say

$$\lim_{x \to p} f(x) = q$$

if $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $0 < d(x,p) < \delta \Rightarrow d(f(x), q) < \varepsilon$.

Ex $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ has $\lim_{x \to p} f(x) = p$ (take $\delta = \varepsilon$)

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$ has $\lim_{x \to 0} f(x) = 0$ (take $\delta =$ anything)

Prop Suppose $E \subset X$, p is a limit point of E, and $f: E \to Y$. Then,

$\lim_{x \to p} f(x) = q \iff$ for all sequences $\{p_n\}$ with $p_n \to p$,
and $p_n \neq p$ for all n,
$f(a_n) \to q$.

Pf $(\Rightarrow)$ Fix any seq. $\{p_n\}$ with $p_n \to p$, $p_n \neq p$ for all n.
Fix any $\varepsilon > 0$. Then, $\exists \delta > 0$ s.t. $0 < d(x,p) < \delta \Rightarrow d(f(x), q) < \varepsilon$.
Since $p_n \to p$ and all $p_n \neq p$, $\exists N$ s.t. $n \geq N \Rightarrow 0 < d(p_n, p) < \delta$.
Thus $n \geq N \Rightarrow d(f(p_n), q) < \varepsilon$.
This says precisely that $f(p_n) \to q$.

$(\Leftarrow)$ Prove the contrapositive: suppose $\lim_{x \to p} f(x) \neq q$. Then $\exists \varepsilon > 0$ s.t.
$\forall n \in \mathbb{N}$, $\exists p_n \in E$ with $d(f(p_n), q) \geq \varepsilon$ and $d(p_n, p) < \frac{1}{n}$.
This gives a sequence $\{p_n\}$ with $p_n \to p$ but not $f(p_n) \to q$. $\blacksquare$

Cor If $\lim_{x \to p} f(x) = q$ and $\lim_{x \to p} f(x) = q'$ then $q = q'$.

Prop Say X metric space, $E \subset \mathbb{R}$, $f,g: E \to \mathbb{R}$,

$$\lim_{x\to p} f(x) = A, \quad \lim_{x \to p} g(x) = B.$$

Then a) $\lim_{x\to p} (f+g)(x) = A+B$

b) $\lim_{x\to p} (fg)(x) = AB$

c) if $B \neq 0$, $\lim_{x\to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$

__Pf__ Use the previous proposition and the analogous theorem already proven for <u>sequences</u>. ∎

__Def__ Say X, Y metric spaces, $E \subset X$, $p \in E$, and $f: E \to Y$.

Then <u>f is continuous at p</u> if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$$\forall x \in E \quad d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon.$$

<u>f is continuous</u> (on E) if f is continuous at every $x \in E$.

__Ex__ $f: \mathbb{R} \to \mathbb{R}$, 1) $f(x) = x$ is continuous (take $\delta = \varepsilon$)

2) $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ is <u>not</u> continuous at $x = 0$ (if $\varepsilon = 1/2$, <u>no</u> δ works)

3) $f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ x & \text{if } x \in \mathbb{Q} \end{cases}$ is continuous <u>only</u> at $x = 0$

__Rk__ If p is not a limit pt of E, then f is always continuous at p.
(Because $\exists \delta > 0$ s.t. $x \in E$, $d(x,p) < \delta \implies x = p$.)

Prop Say X, Y metric spaces, $E \subset X$, $p \in E$, and $f: E \to Y$.
If p is a limit point of E, then

$$f \text{ is continuous at } p \iff \lim_{x \to p} f(x) = f(p).$$

__Pf__ Just compare def. of continuity with def. of limit.

Thm Say X, Y, Z metric spaces, $E \subset X$, $f: E \to Y$, $g: f(E) \to Z$, $h: E \to Z$ given by $h(x) = g(f(x))$

If f is continuous at p and g is continuous at $f(p)$ then h is continuous at p.

Pf Say $\varepsilon > 0$.

Then $\exists \eta > 0$ s.t. $d(y, f(p)) < \eta \Rightarrow d(g(y), g(f(p))) < \varepsilon$.

and $\exists \delta > 0$ s.t. $d(x, p) < \delta \Rightarrow d(f(x), f(p)) < \eta$.

Combine these, setting $y = f(x)$, to get

$$d(x,p) < \delta \Rightarrow d(g(f(x)), g(f(p))) < \varepsilon$$

i.e. $d(x,p) < \delta \Rightarrow d(h(x), h(p)) < \varepsilon$. $\blacksquare$