Continuity

**Prop** \( f : X \to Y \) cont \( \iff \) \( \forall \) sequences \( \{p_n\} \) in \( X \) with \( p_n \to p \), \( f(p_n) \to f(p) \).

**Pf**

\( \Rightarrow \quad \lim_{x \to p} f(x) = f(p) \) by continuity, and \( f(p_n) \to \lim_{x \to p} f(x) \).

\( \Leftarrow \quad \) If \( p \) a limit point, already showed the hypothesis implies that \( \lim_{x \to p} f(x) = f(p) \), and that this eq says \( f \) is ch at \( p \).

If \( p \) not a limit pt, then continuity at \( p \) is automatic.

**Thm** \( f : X \to Y \) is continuous \( \iff \forall V \subseteq Y \) open, \( f^\leftarrow(V) \subseteq X \) is open.

**Pf**

\( \Rightarrow \quad \) Say \( V \subseteq Y \) open. Then say \( p \in f^\leftarrow(V) \), i.e. \( p \in X \), \( f(p) \in V \).

\( \forall \) open, \( \exists \delta > 0 \) st. \( N_{\delta}(f(p)) \subseteq V \). \( f(\delta) \), so \( \exists \delta > 0 \) st.

\( f(N_{\delta}(p)) \subseteq N_{\delta}(f(p)) \). Thus \( N_{\delta}(p) \subseteq f^\leftarrow(V) \). So \( f^\leftarrow(V) \) is open.

\( \Leftarrow \quad \) Fix \( p \in X \), \( \varepsilon > 0 \). Let \( V = \{ y \in Y \mid d(y, f(p)) < \varepsilon \} \).

\( f^\leftarrow(V) \) is open and contains \( p \). Thus \( \exists \delta > 0 \) st. \( N_{\delta}(p) \subseteq f^\leftarrow(V) \).

Thus \( f \) is ch at \( p \).

**Cor** \( f : X \to Y \) is continuous \( \iff \forall V \subseteq Y \) closed, \( f^\leftarrow(V) \subseteq X \) is closed.

**Pf** Use \( V \) closed \( \iff V^c \) open, \( f^\leftarrow(V^c) = (f^\leftarrow(V))^c \).

**Thm** \( f, g : X \to \mathbb{R} \) continuous \( \Rightarrow f + g, fg \) continuous.

If also \( g^{-1}(\{0\}) = \emptyset \), then \( f/g \) continuous.

**Pf**

If \( p \in X \) not a limit point, then continuity at \( p \) is automatic.

If \( p \in X \) is a limit point, then \( \lim_{x \to p} f(x) = f(p) \), \( \lim_{x \to p} g(x) = g(p) \).

And we know \( \lim_{x \to p} (f + g)(x) = \lim_{x \to p} f(x) + \lim_{x \to p} g(x) \).

Thus \( \lim_{x \to p} (f + g)(x) = (f + g)(p) \). So \( f + g \) is continuous at \( p \).

Similarly for \( fg, f/g \).

**Thm** \( f : X \to \mathbb{R}^n \) continuous \( \iff \) each component \( f_i \) of \( f = (f_1, \ldots, f_n) \) is continuous.

**Pf** Sketch

Use \( |f(x) - f_i(x)| \leq |f(x) - f(x_i)| \leq \sum_{i=1}^n |f_i(x) - f_i(x_i)| \).

**Ex**

- The identity \( f : \mathbb{R}^n \to \mathbb{R}^n \) is continuous. (take \( \varepsilon = \delta \))
  - Thus also each coordinate map \( \chi_i : \mathbb{R} \to \mathbb{R} \) is continuous, by the above then.
  - Thus each polynomial in the \( \chi_i \) is continuous.
  - Thus each rational function in the \( \chi_i \) is continuous.

**Ex** For any \( p \in X \), the function \( d_p : X \to \mathbb{R} \) given by \( d_p(x) = d(x, p) \) is continue.
**Thm.** $f: X \to Y$ cts, $X$ compact $\Rightarrow f(X)$ compact.

**Pf.** Say $\{V_a\}$ open cover of $f(X)$. Then $\{f^{-1}(V_a)\}$ is open cover of $X$. $X$ compact, so have finite subcover, i.e., $X \subseteq f^{-1}(V_{a_1}) \cup \cdots \cup f^{-1}(V_{a_n})$. But $f(f^{-1}(Z)) \subseteq Z$, so $f(X) \subseteq V_{a_1} \cup \cdots \cup V_{a_n}$. Thus $f(X)$ is compact.

**Cor.** $f: X \to \mathbb{R}^k$ cts and $X$ compact $\Rightarrow f(X)$ closed and bounded.

**Cor.** $f: X \to \mathbb{R}$ cts and $X$ compact $\Rightarrow \exists p,q \in X$ s.t. $f(p) = \sup f(X), f(q) = \inf f(X)$.

**Pf.** $f(X)$ is bounded. But for any bounded $E \subseteq \mathbb{R}$, $\sup E$ is a limit point of $E$. (why?) But $f(X)$ is closed. Thus $\sup f(X)$, i.e. $\exists p \in X$ s.t. $f(p) = \sup f(X)$. Similarly for inf.

**Rk.** If $X$ not compact we can easily violate this - e.g. if $X = (0,1)$ take $f(x) = x$ or $f(x) = \frac{1}{x}$.

**Thm.** Suppose $f: X \to Y$ is continuous and bijective, and $X$ compact.
Then $f^{-1}: Y \to X$ is continuous.

**Pf.** Using bijectivity: $(f^{-1})^{-1}(V) = V$. Thus we just need to show that $V \subseteq X$ open $\Rightarrow f(V) \subseteq Y$ open.
$V^c$ is closed subset of $X$, so $V^c$ is compact. Thus $f(V^c)$ is also compact. Hence $f(V)$ is closed.
$f$ bijective $\Rightarrow f(V^c) = f(V)^c$. Thus $f(V)$ is open.

**Rk.** If $X$ not compact, this can fail too, e.g. if $X = [0,1)$ take $f(x) = \left(\cos 2\pi x, \sin 2\pi x \right)$.

**Def.** $f: X \to Y$ is uniformly continuous if $\forall \varepsilon > 0 \exists \delta > 0$ s.t.

$\forall p,q \in X, \ d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$.

(Difference from the def. of "continuity" is that \( \delta \) is not allowed to depend on \( x \!\).)

**Rk.** $f: (0,1] \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is not uniformly continuous.

Similarly for $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$. (take $\varepsilon = 1, \delta > 0, \ x = \frac{\delta}{2}; \ (x + \frac{\delta}{2})^2 - x^2 = 1 + \frac{\delta^2}{4} > \varepsilon$)

But $f(x) = cx$ is unif cts. (take $\delta = \frac{\varepsilon}{c}$)

**Thm.** $f: X \to Y$ continuous, $K$ compact $\Rightarrow f$ uniformly continuous.

**Pf.** Fix $\varepsilon > 0$. $f$ cts $\Rightarrow \forall \phi \in X$, $\exists \phi(p)$ s.t. $q \in X, \ d_X(p,q) < \phi(p) \Rightarrow d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$.

Let $J(p) = \{q \in X \mid d_X(p,q) < \phi(p)\}$. The collection $\{J(p)\}_{p \in X}$ is an open cover of $X \Rightarrow$ it has a finite subcover $\{J(p_1), \ldots, J(p_n)\}$.

Let $\delta = \frac{1}{2} \min \{\phi(p_1), \ldots, \phi(p_n)\}$. 

New say \( p, q \in X \) and \( d(p, q) < \delta \).
Then \( \exists m \) st. \( d(q, pm) < \frac{1}{2} \varphi(p, m) \). Also
\[
\begin{aligned}
& d_x(q, pm) \leq d_x(p, q) + d_x(p, pm) \\
& < \delta + \frac{1}{2} \varphi(p, m) \\
& \leq \varphi(p, m)
\end{aligned}
\]

So both \( p \) and \( q \) are within \( \varphi(p, m) \) of \( pm \).
Thus
\[
\begin{aligned}
& d_y(f(p), f(q)) \leq d_y(f(p), f(pm)) + d_y(f(pm), f(q)) \\
& < \frac{\varphi}{2} + \frac{\varphi}{2} \\
& = \varepsilon.
\end{aligned}
\]

\[\text{Then } f: X \to Y \text{ continuous, } E \subseteq X \text{ connected } \implies f(E) \subseteq Y \text{ connected.}\]

**Pf** We'll show \( f(E) \) disconnected \( \implies E \) disconnected.

Say \( f(E) = A \cup B \), \( A,B \) separated nonempty.
Then say \( G = E \cap f^{-1}(A) \), \( H = E \cap f^{-1}(B) \), \( E = G \cup H \). Also \( G \cap H \) nonempty.
(since if \( G = \emptyset \) then \( \nexists p \in E \) with \( f(p) \in A \); similar for \( H \).) Need to show: \( G \cap H \) separated.
\( G \subseteq f^{-1}(A) \cap f^{-1}((A)) \). \( f^{-1}(A) \) closed (since \( f \text{cts} \)).
Thus \( G \subseteq f^{-1}(A) \) also. So \( f(A) \subseteq f(f^{-1}(A)) < A \).
But \( f(H) = B \) and \( A \cap B = \emptyset \). Thus \( f(C) \cap f(H) = \emptyset \). So \( G \cap H = \emptyset \).
Similarly \( G \cap H = \emptyset \).

\[\text{Then } f: [a, b] \to \mathbb{R} \text{ cts.}\]
If \( f(a) < c < f(b) \), then \( \exists x \in [a, b] \) s.t. \( f(x) = c \).

**Pf** We showed earlier \([a, b]\) connected.
Thus \( f([a, b]) \) connected.
But we proved: \( E \subseteq \mathbb{R} \) connected, \( x, y \in E \): \( x < z < y \implies z \in E \).

This is sometimes called "Intermediate Value Theorem".