Continuity

Prop $f: X \to Y$ cts $\iff$ $\forall$ sequences $\{p_n\}$ in X with $p_n \to p$, $f(p_n) \to f(p)$.

Pf $(\Rightarrow)$ $\lim_{x\to p} f(x) = f(p)$ by continuity, and $f(p_n) \to \lim_{x \to p} f(x)$. ∎

$(\Leftarrow)$ If p is a limit pt, we already showed the hypothesis implies that $\lim_{x\to p} f(x) = f(p)$, and that this eq says f is cts at p.
If p is not a limit pt, then continuity at p is automatic. ∎

Thm $f: X \to Y$ is continuous $\iff$ $\forall V \subset Y$ open, $f^{-1}(V) \subset X$ is open

Pf $(\Rightarrow)$ Say $V \subset Y$ open. Then say $p \in f^{-1}(V)$, i.e. $p \in X$, $f(p) \in V$.
V open, so $\exists \varepsilon > 0$ s.t. $N_\varepsilon(f(p)) \subset V$. f cts, so $\exists \delta > 0$ s.t.
$f(N_\delta(p)) \subset N_\varepsilon(f(p))$. Thus $N_\delta(p) \subset f^{-1}(V)$. So $f^{-1}(V)$ is open.

$(\Leftarrow)$ Fix $p \in X$, $\varepsilon > 0$. Let $V = \{y \in Y \mid d(y, f(p)) < \varepsilon\}$.
$f^{-1}(V)$ is open and contains p. Thus $\exists \delta > 0$ s.t. $N_\delta(p) \subset f^{-1}(V)$, ie
$f(N_\delta(p)) \subset V$. Thus f is cts at p. ∎

Cor $f: X \to Y$ is continuous $\iff$ $\forall V \subset Y$ closed, $f^{-1}(V) \subset X$ is closed.

Pf Use V closed $\iff V^c$ open, $f^{-1}(E^c) = (f^{-1}(E))^c$. ∎

Thm $f, g: X \to \mathbb{R}$ continuous $\Rightarrow$ $f+g$, fg continuous
if also $g^{-1}(\{0\}) = \phi$, then f/g continuous.

Pf If $p \in X$ not a limit point, then continuity at p is automatic.
If $p \in X$ is a limit point, then $\lim_{x\to p} f(x) = f(p)$, $\lim_{x \to p} g(x) = g(p)$.
And we know $\lim_{x\to p}(f+g)(x) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x)$.
Thus $\lim_{x\to p}(f+g)(x) = (f+g)(p)$. So $f+g$ is continuous at p.
Similarly for fg, f/g. ∎

Thm $f: X \to \mathbb{R}^n$ continuous $\iff$ each component f_i of $f = (f_1, \dots, f_n)$ is continuous.

Pf Sketch Use $|f_i(x) - f_i(x')| \le |f(x) - f(x')| \le \sum_{i=1}^n |f_i(x) - f_i(x')|$. ∎

Ex
- The identity $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuous. (take $\varepsilon = \delta$)
- Thus also each coordinate map $x_i: \mathbb{R}^n \to \mathbb{R}$ is continuous, by the above thm.
- Thus each polynomial in the x_i is continuous.
- Thus each rational function in the x_i is continuous.

Ex For any $p \in X$, the function $d_p: X \to \mathbb{R}$ given by $d_p(x) = d(x, p)$ is continuous.

Thm $f: X \to Y$ cts, X compact $\Rightarrow f(X)$ compact.

Pf Say $\{V_\alpha\}$ open cover of $f(X)$. Then $\{f^{-1}(V_\alpha)\}$ is open cover of X.
X compact, so have finite subcover, ie, $X \subset f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n})$.
But $f(f^{-1}(Z)) \subset Z$, so $f(X) \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$.
Thus $f(X)$ is compact. ∎

Cor $f: X \to \mathbb{R}^k$ cts and X compact $\Rightarrow f(X)$ closed and bounded.

Cor $f: X \to \mathbb{R}$ cts and X compact $\Rightarrow \exists\, p,q \in X$ s.t. $f(p) = \sup f(X)$, $f(q) = \inf f(X)$.

Pf $f(X)$ is bounded. But for any bounded $E \subset \mathbb{R}$, $\sup E$ is a limit point of E. (why?)
But $f(X)$ is closed. Thus $\sup f(X) \in f(X)$, i.e. $\exists\, p \in X$ s.t. $f(p) = \sup f(X)$.
Similarly for inf. ∎

Rk If X not compact we can easily violate this — e.g. if $X = (0,1]$ take $f(x) = x$ or $f(x) = \frac{1}{x}$.

Thm Suppose $f: X \to Y$ is continuous and bijective, and X compact.
Then $f^{-1}: Y \to X$ is continuous.

Pf Using bijectivity: $(f^{-1})^{-1}(V) = V$. Thus, we just need to show that $V \subset X$ open $\Rightarrow f(V) \subset Y$ open.
V^c is closed subset of X, so V^c is compact. Thus $f(V^c)$ is also compact. Hence $f(V^c)$ is closed.
f bijective $\Rightarrow f(V^c) = f(V)^c$. Thus $f(V)$ is open.

Rk If X not compact, this can fail too, e.g. if $X = [0,1)$ take $f(x) = (\cos 2\pi x, \sin 2\pi x)$

Def $f: X \to Y$ is <u>uniformly continuous</u> if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$\forall p,q \in X,\ d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon.$$

(Difference from the def. of "continuous" is that δ is <u>not</u> allowed to depend on x!)

Rk $f: (0,1] \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is not uniformly continuous.
Similarly for $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$. (take $\varepsilon = 1$, $\delta > 0$, $x = \frac{1}{\delta}$; $(x + \frac{\delta}{2})^2 - x^2 = 1 + \frac{\delta^2}{4} > \varepsilon$)
But $f(x) = cx$ <u>is</u> unif cts. (take $\delta = \frac{\varepsilon}{c}$)

Thm $f: X \to Y$ <u>continuous</u>, K compact $\Rightarrow f$ <u>uniformly continuous</u>.

Pf Fix $\varepsilon > 0$. f cts $\Rightarrow \forall p \in X$, $\exists \phi(p)$ s.t. $q \in X$, $d_X(p,q) < \phi(p) \Rightarrow d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$.
Let $J(p) = \{q \in X \mid d_X(p,q) < \frac{1}{2}\phi(p)\}$. The collection $\{J(p)\}_{p \in X}$ is an open cover of $X \Rightarrow$ it has a finite subcover $\{J(p_1), \ldots, J(p_n)\}$.

Let $\delta = \frac{1}{2}\min\{\phi(p_1), \ldots, \phi(p_n)\}$.

Now say $p, q \in X$ and $d(p,q) < \delta$.

Then $\exists m$ s.t. $d_X(p, p_m) < \frac{1}{2}\phi(p_m)$. Also

$$\begin{aligned} d_X(q, p_m) &\leq d_X(p,q) + d_X(p, p_m) \\ &< \delta + \tfrac{1}{2}\phi(p_m) \\ &\leq \phi(p_m) \end{aligned}$$

So both p and q are within $\phi(p_m)$ of p_m.

Thus

$$\begin{aligned} d_Y(f(p), f(q)) &\leq d_Y(f(p), f(p_m)) + d(f(p_m), f(q)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

■

Thm $f: X \to Y$ continuous, $E \subset X$ connected $\Rightarrow f(E) \subset Y$ connected.

Pf We'll show $f(E)$ disconnected $\Rightarrow E$ disconnected.

Say $f(E) = A \cup B$, A, B separated nonempty

Then say $G = E \cap f^{-1}(A)$, $H = E \cap f^{-1}(B)$. $E = G \cup H$. Also G, H nonempty (since if $G = \emptyset$ then $\nexists\, p \in E$ with $f(p) \in A$; similar for H). Need to show: G, H separated.

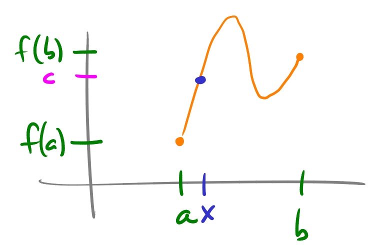
$G \subset f^{-1}(A) \subset f^{-1}(\bar{A})$. $f^{-1}(\bar{A})$ closed (since f cts).

Thus $\bar{G} \subset f^{-1}(\bar{A})$ also. So $f(\bar{G}) \subset f(f^{-1}(\bar{A})) \subset \bar{A}$.

But $f(H) = B$ and $\bar{A} \cap B = \emptyset$. Thus $f(\bar{G}) \cap f(H) = \emptyset$. So $\bar{G} \cap H = \emptyset$.

Similarly $G \cap \bar{H} = \emptyset$. ■

Thm Say $f: [a,b] \to \mathbb{R}$ cts.
If $f(a) < c < f(b)$, then $\exists x \in [a,b]$ s.t. $f(x) = c$.



Pf We showed earlier $[a,b]$ connected.

Thus $f([a,b])$ connected.

But we proved: $E \subset \mathbb{R}$ connected, $x, y \in E$, $x < z < y \Rightarrow z \in E$. ■

This is sometimes called "Intermediate Value Theorem".