Exercise 1. Guillemin/Pollack Chapter 1, §4 (p. 25): 2, 5, 9, 10. For 9 and 10 you need the definition of the orthogonal group: it is the group of all $n \times n$ real matrices $A$ obeying $AA^T = 1$ (as discussed on pages 22-23).

Exercise 2. This exercise is essentially a tautology, which we already mentioned in class, but which seems worth a moment’s reflection.

Let $M$ be a smooth manifold, $U \subset M$ an open subset, and $x : U \to \mathbb{A}^n$. Prove that $(U, x)$ is a chart if and only if $x : U \to f(U)$ is a diffeomorphism.

Exercise 3. Let $M$ be a manifold. The tangent bundle $TM$ is defined as the disjoint union of all the tangent spaces, $\bigcup_{p \in M} T_pM$. Equip $TM$ with a natural topology and smooth atlas. (Hint: Consider a chart $(U, x)$ of $M$. Then, essentially by our definition of $T_pM$, we have a natural identification between the open subset $\bigcup_{p \in U} T_pM \subset TM$ and $U \times \mathbb{R}^n$.) What is the dimension of $TM$? Can you describe $TS^1$ as a familiar manifold?

Exercise 4. This exercise gives a little more practice working with projective spaces.

1. Define complex projective space $\mathbb{CP}^n$ as the set of equivalence classes $[z^0, z^1, \ldots, z^n] : (z^0, z^1, \ldots, z^n) \neq (0, 0, \ldots, 0) \mod \sim$, where $[z^0, \ldots, z^n] \sim [z'^0, \ldots, z'^n]$ if and only if $z'^i = \lambda z^i$ for some $\lambda \in \mathbb{C}^\times$. Put a natural structure of smooth manifold on $\mathbb{CP}^n$. (Consider $U_i = \{[z^0, \ldots, z^n] : z^i \neq 0\}$.) Construct a diffeomorphism between $\mathbb{CP}^1$ and the standard 2-sphere.

2. Suppose you identify the 3-sphere with the unit sphere in $\mathbb{C}^2$ $S^3 = \{(z^1, z^2) \in \mathbb{C}^2 : |z^1|^2 + |z^2|^2 = 1\}$.

Then show that the map $f : S^3 \to S^2$

$f(z^1, z^2) \mapsto [z^1, z^2]$

is a submersion. What is the inverse image of a point? The map $f$ is called the Hopf fibration. (For fun, think about what you can say about the inverse image of a 2-point subset of $S^2$.)

Exercise 5. Here is a more abstract, coordinate-free approach to the manifold structures on projective spaces. Let $V$ be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ and denote by $PV$ the set of lines in $V$. Recall that a line is a 1-dimensional vector space, so a line in $V$ is a one-dimensional subspace of $V$. 


Let $L \subset V$ be a line and $W \subset V$ a complementary subspace, i.e., $V = L \oplus W$. Define
\[ \phi_{L,W} : \text{Hom}(L, W) \rightarrow PV \]
\[ T \mapsto L_T \]
where $L_T = \{T + \ell \ell : \ell \in L\}$ is the graph of $T$. We can identify $L_T$ as the image of the linear map $1_L + T : L \rightarrow L \oplus W = V$. Show that the image of $\phi_{L,W}$ is $PV \setminus PW$.

Now consider a second pair $(L', W')$ and the corresponding $\phi_{L',W'}$. We now have two parametrizations of $PV \setminus (PW \cup PW')$, so can compare by an overlap isomorphism
\[ f : U \rightarrow U', \]
where $U \subset \text{Hom}(L, W)$ and $U' \subset \text{Hom}(L', W')$ are the images of $PV \setminus (PW \cup PW')$ under the two parametrizations. Write a formula for the map $f$. Show that $f$ is smooth. (Hint: The formula involves $\pi^{L',W'} : V \rightarrow L'$, the projection onto $L'$ with kernel $W'$.)

Use the parametrizations $\phi_{L,W}$ to topologize $PV$ and construct a smooth atlas, so make $PV$ a smooth manifold. What is its dimension? Prove that an injective linear map $V' \rightarrow V$ induces a smooth map $PV' \rightarrow PV$.

Construct a natural isomorphism (take this to mean that it doesn’t depend on choices)
\[ T_L(PV) \rightarrow \text{Hom}(L, V/L). \]

**Exercise 6.** This exercise is preparation for our discussion of partitions of unity.

1. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by
\[ f(x) = \begin{cases} e^{-1/x^2}, & x > 0; \\ 0, & x \leq 0. \end{cases} \]
Prove that $f$ is $C^\infty$. Sketch the graph of $f$. Compare $f$ to its Taylor series at $x = 0$.

2. Given real numbers $a < b$ show that
\[ g(x) := f(x - a)f(b - x) \]
is smooth and vanishes outside the interval $(a, b)$.

3. Given real numbers $a < b$, construct a $C^\infty$ function $h$ such that: (i) $h(x) = 0$ for $x \leq a$, (ii) $h(x) = 1$ for $x \geq b$, and (iii) $h$ is monotonic nondecreasing.

4. Given real numbers $a < b < c < d$, construct a $C^\infty$ function $k$ so that (i) $k(x) = 0$ for $x \leq a$, (ii) $k(x) = 1$ for $b \leq x \leq c$, and (iii) $k(x) = 0$ for $x \geq d$.

5. Given real numbers $a^i < b^i < c^i < d^i$, $i = 1, \ldots, n$, construct a $C^\infty$ function $k : \mathbb{A}^n \rightarrow \mathbb{R}$ so that (i) $k(x^1, \ldots, x^n) = 0$ if any $x^i \leq a^i$; (ii) $k(x^1, \ldots, x^n) = 1$ if $b^i \leq x^i \leq c^i$ for all $i = 1, \ldots, n$; and (iii) $k(x^1, \ldots, x^n) = 0$ if any $x^i \geq d^i$.

6. Prove that on every manifold $M$ there is a nonconstant $C^\infty$ function $f : M \rightarrow \mathbb{R}$. 
