**Def** \[ M \text{ a topological space:} \]

1) \( M \text{ is locally Euclidean if, } \forall p \in M, \exists \text{ a open } U \subset M \text{ and a homeomorphism } \phi : U \rightarrow V \text{ where } V \subset \mathbb{A}^n \text{ open.} \)

2) \( M \text{ is topological manifold if } M \text{ is locally Euclidean, Hausdorff, and has a countable basis.} \)

**Ex** \( M = S^1 \text{ is locally Euclidean: } \) every point has a neighborhood homeomorphic to an open interval \( V \subset \mathbb{A}^1 \).

Also Hausdorff and has countable base.

union of disc and line is not locally Euclidean at \( p \).

This follows from

**Thm (Invariance of Domain)** \( \text{If } V_1 \subset \mathbb{A}^n, \ V_2 \subset \mathbb{A}^n \text{ both open and } V_1 \cong V_2 \text{ (homeo), then } n_1 = n_2. \)

**Pf** Not easy! Omitted here. (Standard way uses homology)

**Cor** \( \text{If } M \text{ is locally Euclidean, there is a unique, locally constant function } \dim_M : M \rightarrow \mathbb{Z}_{\geq 0} \) such that, for \( p \in M, \exists \text{ a open } U \text{ of } p \text{ with } U \cong V \subset \mathbb{A}^{\dim_M(p)}. \)

**Pf** Fix \( n, \) show \( \dim_M^n(n) \) open and closed in \( M. \)

**Ex** "Line with doubled origin" \( \mathbb{A}^1 \cup \mathbb{A}^1 \setminus \{0\} \)

is locally Euclidean but not Hausdorff.

**Ex** Uncountable set with discrete topology is locally Euclidean \( (\dim = 0) \)

but has no countable basis.
Def 1) A topological manifold: a chart on $M$ is a pair $(U, x)$ with
- $U \subset M$ open
- $x: U \to \mathbb{A}^n$ homeomorphism into $x(U)$, with $x(U) \subset \mathbb{A}^n$ open

2) Charts $(U, x)$ and $(V, y)$ are $C^\infty$-related if $x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$ and its inverse $x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$ are both $C^\infty$ maps.

Ex Let $S^2 \subset \mathbb{A}^3$ be the set $\{(a, b, c) : a^2 + b^2 + c^2 = 1\}$

$U_1 = \{a > 0\} \subset S^2$ \hspace{1cm} $U_1 \cap U_2 = \{a > 0, b > 0\}$ \hspace{1cm} $U_2 = \{b > 0\} \subset S^2$

$(U_1; (a, b, c))$ and $(U_2; (a, b, c))$ are charts.
Each has image a disc in $\mathbb{A}^2$.

Overlap map: $(a, c) \mapsto (b = \sqrt{1 - a^2 - c^2}, c)$
Inverse map: $(b, c) \mapsto (a = \sqrt{1 - b^2 - c^2}, c)$
both smooth

so these charts are $C^\infty$-related.

Rk
- These 2 charts don't cover $S^2$ — would need 6 similar charts to do this.
- In HW, make a covering of $S^2$ by 2 $C^\infty$-related charts.
- A covering of $S^2$ by a single chart. Indeed, if $(S^2, x)$ is a chart then $x(S^2)$ is open in $\mathbb{A}^2$ and also compact.
- Can similarly cover $S^n = \{(x_1)^2 + \cdots + (x_{n+1})^2 = 1\} \subset \mathbb{A}^{n+1}$ by $2(n+1)$ charts.
**Def** M topological manifold: a smooth atlas on M is a collection \( C = \{(U, x)\}_{\alpha \in A} \) of charts such that

1) \( \bigcup_{\alpha \in A} U_\alpha = M \)

2) If \( \alpha_1, \alpha_2 \in A \) then \((U, x)\) and \((U', x')\) are \( C^\infty \)-related

3) \( C \) is maximal wrt 1), 2)

**Prop** Any \( C \) obeying 1), 2) can be uniquely extended to a smooth atlas.

**Pf** The extended atlas consists of all \((U, x)\) which are \( C^\infty \)-related to every \( x \) in \( C \). (Check this atlas still satisfies 2)!

**Ex**

1) \( \{(A', x)\} \) completes to a smooth atlas on \( A' \).

2) \( \{(A', x')\} \) "a different smooth atlas \( C' \) on \( A' \) with \( x \not\in C' \).

3) \( \{(A'', x)\} \) completes to a smooth atlas on \( A'' \). ("standard")

4) \( \{6 \text{ charts discussed above on } S^3\} \) completes to a smooth atlas on \( S^2 \).

**Def** A smooth manifold is a pair \((M, C)\) : \( M \) a topological manifold, \( C \) a smooth atlas on \( M \).

Usually don't write \( C \) explicitly, use "smooth chart on \( M \)" to mean "chart in \( C \)."

**Def** \( M, N \) smooth manifolds, \( f: M \to N \):

1) \( p \in M \) : \( f \) is smooth at \( p \) if, \( \forall \text{ smooth charts } (U, x) \text{ on } M \) with \( p \in U \)

\[ y \circ f \circ x^{-1}: x(U \cap f^{-1}(V)) \to y(V) \text{ is smooth at } x(p). \]

2) \( f \) is smooth if \( \forall p \in M \) \( f \) is smooth at \( p \).

3) \( f \) is diffeomorphism if \( f \) is bijective and both \( f \) and \( f^{-1} \) are smooth.

Fortunately we don't really have to check all charts:

**Prop** \( f \) is smooth at \( p \) \( \iff \exists \) smooth charts \( (U, x) \text{ on } M \) with \( p \in U \)

\[ y \circ f \circ x^{-1} \text{ smooth at } x(p). \]

**Pf** If \((U', x')\) is another chart on \( M \) and \((V', y')\) another on \( N \),

then \( y' \circ f' \circ x'^{-1} = (y' \circ y) \circ (y \circ f \circ x^{-1}) \circ (x \circ x'^{-1}) \) composition of smooth functions.
The antipodal map $S^2 \to S^2$ is smooth: it takes $(a,b) \to (-a,-b)$.

A given topological manifold may admit many non-diffeomorphic smooth atlases; e.g., $M = \mathbb{R}^4$ is like this (but not $\mathbb{A}^n$ for any $n \neq 4$!) or it may admit none at all.

Guillemin-Pollack def of manifold involves taking $M$ to be $\subset \mathbb{A}^N$ for some $N$. We'll prove later that theirs is equivalent to our def.

1) If $M$ is a smooth mfd and $U \subset M$ open, then $U$ is smooth mfd. (HW)
2) $GL_n \mathbb{R} = \{ n \times n \text{ matrices } A \text{ with } \det A \neq 0 \}$ is open in $\mathbb{R}^{n^2}$, hence is smooth mfd; similarly $GL_n \mathbb{C}$ is open in $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$
3) If $M, N$ are smooth mfds then $M \times N$ is a smooth mfd. (HW)