This room is not a vector space: what does it mean to "add" two points?

The precise structure it has is well captured by the notion of affine space.

**Def** A vector space (over \( \mathbb{R} \)): an affine space \( A \) over \( V \) is a set with a free transitive action of \( V \).

Think of \( x \in A \) as points, \( \vec{x} \in V \) as arrows.

Write the action as \( + : A \times V \to A \) \( x \in A, \vec{x} \in V \mapsto x + \vec{x} \in A \)

**Operations**

\[
\begin{align*}
+ & : V \times V \to V \\
- & : V \times V \to V \\
\cdot & : \mathbb{R} \times V \to V
\end{align*}
\]

Standard example:

\[
\begin{align*}
V &= \mathbb{R}^n = \{(\vec{x}_1, \ldots, \vec{x}_n) : \vec{x}_i \in \mathbb{R}\} \\
A &= \mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}
\end{align*}
\]

\( + : A \times V \to A \) is componentwise addition

**Derivative**

Say \( A \) affine space over \( V \), \( A' \) affine space over \( V' \), \( U \subset A \) open, \( f : U \to A' \), \( p \in U \), \( \vec{x} \in V \).

**Def** (Directional derivative) \( \vec{x} f(p) = \lim_{t \to 0} \frac{f(p+t\vec{x})-f(p)}{t} \in V' \) (if this limit \( \exists \))
Thm/Def  If \( \exists f(p) \) exists \( \forall p \in U, \exists \in V \)
and \( \exists f: U \rightarrow V' \) is continuous \( \forall \exists \in V \)
then \( \exists f(p) \) is a linear funcn of \( \exists \).

It is denoted \( df_p \).

Pf  On choosing a basis, this becomes a standard statement from multivariate calculus.

\[ df_p : V \rightarrow V', \text{ ie } df_p \in \text{Hom}(V, V') \]
\[ \text{or, } df : U \rightarrow \text{Hom}(V, V') \]

Def  \( f \) is smooth in \( U \) if \( \forall \exists \in \mathbb{R} \) and \( \exists_1, \ldots, \exists_5 \in V, \exists_1, \ldots, \exists_5 f \in V' \) exists.

Thm  If \( f \) is smooth on \( U \), and \( \exists, \exists_2 \in V \), then \( \exists, \exists f = \exists_2, \exists f \).

Pf  On choosing a basis, this follows from the equality of mixed partials.

Thm  If \( f : A \rightarrow A' \) and \( g : A' \rightarrow A'' \) smooth
then \( g \circ f \) is smooth and \( [d(g \circ f)]_p = (dg) \circ (df)_p \)

Pf  On choosing a basis, this is the multivariable chain rule.