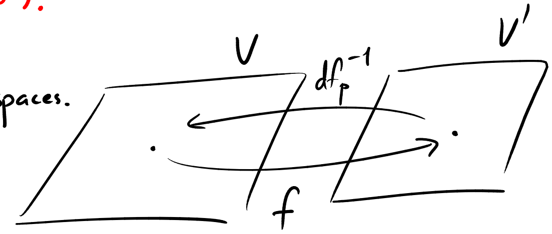
Thm A, A' affine, $U \subset A$, $f: U \to A'$ smooth, $p \in U$, df_p bijective:
$\exists\, \tilde{U} \subset U$ open, $p \in \tilde{U}$, s.t. $f|_{\tilde{U}}$ is a diffeo onto $f(\tilde{U})$.

Pf • Use $p, f(p)$ as origins to identify $A \simeq V$, $A' \simeq V'$ vector spaces.



Compose with df_p^{-1} to reduce to case of

$$f: V \to V \text{ smooth, } df_0 = \mathbb{1}.$$

• Set $\phi(\xi) = \xi - f(\xi)$. Then $\phi(0) = 0$ and $d\phi_0 = \mathbb{1} - \mathbb{1} = 0$. Fix any norm on V.
Choose $\varepsilon > 0$ s.t. $\|d\phi_\xi\| < \frac{1}{2}$ for $\|\xi\| \le \varepsilon$. Then $\|\phi(\xi)\| < \frac{\|\xi\|}{2}$ for $\|\xi\| \le \varepsilon$
i.e. ϕ maps $\overline{B}_\varepsilon \to \overline{B}_{\varepsilon/2}$.

• Say $\eta \in \overline{B_{\varepsilon/2}}$. Want to show $f(\xi) = \eta$ has at most 1 solution near $\xi = 0$.
Moral idea: solve by iteration — approximating f by the identity, we can "update" our guess by taking $\xi \mapsto \xi + (\eta - f(\xi))$ to get a better guess.

So, let $\phi_\eta(\xi) = \eta + \xi - f(\xi)$. Then $f(\xi) = \eta \iff \phi_\eta(\xi) = \xi$.

- $\phi_\eta(\overline{B}_\varepsilon) \subset \overline{B}_\varepsilon$ (by Δ ineq, using $\phi_\eta(\xi) = \eta + \phi(\xi)$)
- $\phi_\eta(\xi_1) - \phi_\eta(\xi_2) = \phi(\xi_1) - \phi(\xi_2) \le \frac{1}{2}\|\xi_1 - \xi_2\|$

Thus ϕ_η is contraction on $\overline{B_\varepsilon}$ $\Rightarrow$ has unique fixed point $\xi \in \overline{B_\varepsilon}$

This ξ is the unique solution of $f(\xi) = \eta$ in $\overline{B_\varepsilon}$.
Thus let $\tilde{U} = B_\varepsilon \cap f^{-1}(B_{\varepsilon/2})$. $f(\tilde{U}) \subset B_{\varepsilon/2}$, $\tilde{U}$ is open, $0 \in \tilde{U}$, and we have constructed $g: f(\tilde{U}) \to \tilde{U}$ with $g \circ f = \mathbb{1}$.

$$\begin{tikzcd} \tilde{U} \arrow[r, shift left, "f"] & f(\tilde{U}) \arrow[l, shift left, "g"] \end{tikzcd}$$
ξ $\qquad$ η

Next, need to show g is smooth.
First: if $\eta_1, \eta_2 \in B_{\varepsilon/2}$ with $g(\eta_i) = \xi_i \in \tilde{U}$

then $\xi_2 - \xi_1 = \eta_2 - \eta_1 + \phi(\xi_2) - \phi(\xi_1)$

so $\|\xi_2 - \xi_1\| \le \|\eta_2 - \eta_1\| + \|\phi(\xi_2) - \phi(\xi_1)\|$

$$\le \|\eta_2 - \eta_1\| + \tfrac{1}{2}\|\xi_2 - \xi_1\|$$

so $\|\xi_2 - \xi_1\| \le 2\|\eta_2 - \eta_1\|$ Thus g is continuous (even Lipschitz)

Next show g once differentiable:

$$\|g(\eta_2)-g(\eta_1)-df^{-1}_{g(\eta_1)}(\eta_2-\eta_1)\| = \|\xi_2-\xi_1-df^{-1}_{\xi_1}(\eta_2-\eta_1)\|$$
$$\leq \|df^{-1}_{\xi_1}\|\cdot\|df_{\xi_1}(\xi_2-\xi_1)-(\eta_2-\eta_1)\|$$
$$= \|df^{-1}_{\xi_1}\|\cdot\|f(\xi_2)-f(\xi_1)-df_{\xi_1}(\xi_2-\xi_1)\|$$

Given $\varepsilon>0$, choose $\delta>0$ s.t. $\|\xi_2-\xi_1\|<\delta \Rightarrow \|f(\xi_2)-f(\xi_1)-df_{\xi_1}(\xi_2-\xi_1)\| < \frac{\varepsilon}{2\|df^{-1}_{\xi_1}\|}\|\xi_2-\xi_1\|$

(such δ $\exists$ since f is smooth, hence once differentiable)

Then if $\|\eta_2-\eta_1\|<\frac{\delta}{2}$, have $\|\xi_2-\xi_1\|<\delta$, and so $\|g(\eta_2)-g(\eta_1)-df^{-1}_{g(\eta_1)}(\eta_2-\eta_1)\| \leq \frac{\varepsilon}{2}\|\xi_2-\xi_1\| \leq \varepsilon\|\eta_2-\eta_1\|$

Thus g is once differentiable, with $dg_\eta = df^{-1}_{g(\eta)}$

Then smoothness follows from smoothness of f and of the map $T\mapsto T^{-1}$. ∎

Cor (Inverse Function Thm) $f: M\to N$ smooth, df_p bijective:
$\exists\, U\subset M$ open with $p\in U$, s.t. $f: U\to f(U)$ is diffeomorphism.

Pf Use the Thm and local coordinate charts, along with facts

1. coordinate charts are diffeomorphisms, (check!)
2. composition of diffeomorphisms is a diffeomorphism. ∎

Def $f: M\to N$ smooth:

1) f is an immersion at p if df_p is injective
2) f is a submersion at p if df_p is surjective
3) f is a local diffeomorphism at p if df_p is bijective

Prop M smooth manifold, $p\in M$, $f: M\to N$:

1) If f is local diffeo at p, then $\exists$ charts $(\overset{p}{\cap}\!U, x)$ and $(\overset{f(p)}{\cap}\!V, y)$ s.t. $y\circ f\circ x^{-1}$ = identity restr. to $x(U$

2) ——— immersion ——————————— inclusion $\mathbb{A}^m\to\mathbb{A}^n$ —

3) ——— submersion ——————————— projection $\mathbb{A}^m\to\mathbb{A}^n$ —

Pf 1) By inverse function thm, can find a chart (U,x) s.t. $f|_U$ is diffeo.
Then take $V=f(U)$ and $y=x\circ f^{-1}$.

2) Fix charts $(\tilde{U}, x)$ and $(\tilde{V}, \tilde{y})$ s.t. $f(\tilde{U}) \subset \tilde{V}$,

$$\begin{tikzcd} \tilde{U} \arrow[r, "f"] \arrow[d, "x"'] & \tilde{V} \arrow[d, "\tilde{y}"] \\ \mathbb{A}^m \arrow[r] & \mathbb{A}^n \end{tikzcd}$$

$$d(\tilde{y} \circ f \circ x^{-1})_{x(p)} = \begin{pmatrix} \mathbb{1}_{m \times m} \\ 0_{n-m \times m} \end{pmatrix} \longleftarrow$$ (can always arrange this by linear coord change)

Then, define $g : \tilde{U} \times \tilde{Z} \to \tilde{V}$, $\tilde{Z} \subset \mathbb{R}^{n-m}$, by $g(p, z) = \tilde{y}^{-1}\left(\tilde{y}(f(p)) + (0, z)\right)$

(take $\tilde{Z}$ small enough that the RHS is well defined)

Now $d(\tilde{y} \circ f \circ (x, \mathbb{1})^{-1})_{(x(p), 0)} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$, so g is local diffeo at $(p, 0) \in \tilde{U} \times \tilde{Z}$

Inverse Function Thm $\Rightarrow$ can pick $U \subset \tilde{U}$, $Z \subset \tilde{Z}$ s.t. g is diffeo on $U \times Z$; set $V = g(U \times Z)$.

Then take $y = (x \circ \pi_1 \oplus \pi_2) \circ g^{-1}$

[ie if $\tilde{y}(v) = \tilde{y}(f(p)) + (0, z)$ then $y(v) = x(p) \oplus z$]

$$\begin{tikzcd} & U \times Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ U & & Z \end{tikzcd}$$

3) Fix charts $(\tilde{U}, \tilde{x})$ and $(\tilde{V}, y)$ s.t. $f(\tilde{U}) \subset \tilde{V}$,

$$d(y \circ f \circ \tilde{x}^{-1}) = \begin{pmatrix} \mathbb{1} & 0 \end{pmatrix}$$

$$\begin{tikzcd} & \tilde{U} \subset \mathbb{A}^m \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{A}^n & & \mathbb{A}^{m-n} \end{tikzcd}$$

Define $g : \tilde{U} \to \tilde{V} \times \mathbb{A}^{m-n}$ by $g(p) = \left(f(p), \pi_2(\tilde{x}(p))\right)$

Then g is a local diffeo. at p.

Inverse Function Thm $\Rightarrow$ can pick $U \subset \tilde{U}$ s.t. g is diffeo on U.

Then define $x = (y \circ \sigma_1 \oplus \sigma_2) \circ g$

ie $x(p) = y(f(p)) \oplus \pi_2(\tilde{x}(p))$

$$\begin{tikzcd} & \tilde{V} \times \mathbb{A}^{m-n} \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ \tilde{V} & & \mathbb{A}^{m-n} \end{tikzcd}$$

<u>Rk</u> Conditions "f is immersion" or "f is submersion" can both be expressed as $\operatorname{rank}(df) = \min(\dim M, \dim N)$ — the max. possible value.

Lemma $S = \{ T \in \mathrm{Hom}(V,W) \mid \mathrm{rank}\, T = \min(\dim V, \dim W) \}$ is open in $\mathrm{Hom}(V,W)$.

Pf If $\dim V = \dim W$ then fix some $\iota : V \xrightarrow{\sim} W$, then define
$$\det : \mathrm{Hom}(V,W) \to \mathbb{R}, \quad T \mapsto \det(T \circ \iota^{-1})$$

$S = \det^{-1}(\mathbb{R} \setminus \{0\})$, $\det$ is continuous, so S is open.

If $\dim V < \dim W$ then fix some $T_0 : V \hookrightarrow W$, and Y s.t. $T_0(V) \oplus Y = W$.

$$p : \mathrm{Hom}(V,W) \to \mathrm{Hom}(V, W/Y), \quad T \mapsto \pi \circ T \qquad \pi : W \to W/Y$$

Then $p^{-1}(\mathrm{Iso}(V, W/Y))$ is open, contains T_0.

[ie: can find basis where $T_0 = \left(1 \mid 0 \right)$ and under small pert, this minor is still nonzero]

If $\dim V > \dim W$ then fix some $T_0 : V_0 \twoheadrightarrow W$ and $V_0 \subset V$ s.t. $T_0 : V_0 \xrightarrow{\sim} W$.

$$p : \mathrm{Hom}(V,W) \to \mathrm{Hom}(V_0, W), \quad T \mapsto T|_{V_0}$$

Then $p^{-1}(\mathrm{Iso}(V_0, W))$ is open, contains T_0. $\blacksquare$

Cor $\{ p \in M \mid \mathrm{rank}(df_p) = \min(\dim M, \dim N) \}$ is open in M.