Def $f: M \to N$ smooth: $p \in M$ is called a <u>critical point</u> if f is not a submersion at p.
$q \in N$ is a <u>critical value</u> if $\exists\, p \in f^{-1}(q)$ s.t. p is a critical point.
$q \in N$ is a <u>regular value</u> if not a critical value.

Ex $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2 + 2$ — $df = [2x]$

f is a submersion everywhere <u>except</u> $x = 0$
thus 2 is the only critical value

$f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = xy$ — $df = [y \;\; x]$

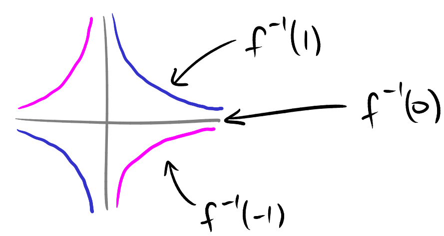
f is a submersion everywhere <u>except</u> $(x,y) = (0,0)$
thus 0 is the only critical value

Rk 1) If $c \notin f(M)$ then c is a regular value (vacuously!)
2) If $\dim(M) < \dim(N)$ then every $c \in f(M)$ is a critical value.
3) We've already shown the set of <u>critical points</u> of f is <u>closed</u> in M.

Prop If c is a regular value of $f: M \to N$, then $f^{-1}(c)$ is a smooth submanifold of M, with dimension $\dim(M) - \dim(N)$.

Ex $f(x,y) = xy$ — then for any $c \neq 0$, $f^{-1}(c)$ is a smooth submanifold of $\mathbb{R}^2$, of dimension $2 - 1 = 1$

[but $f^{-1}(0)$ is <u>not</u>!]



Pf Say c is a regular value, take any $p \in f^{-1}(c)$.
Thm on local behavior of submersions says that we can find charts (x, U) around p and (y, V) around c such that $y \circ f \circ x^{-1}$ is projection $\mathbb{A}^n \to \mathbb{A}^m$ [restricted to $x(U)$] and $x(c) = 0 \in \mathbb{A}^m$.
Thus $x \circ f^{-1}(c) = \{ x^{m+1} = \cdots = x^n = 0 \} \subset x(U)$, i.e. (x, U) is the desired chart. ∎

Rk This proposition is a very useful way of detecting manifolds!

e.g. $f: \mathbb{R}^n \to \mathbb{R}$ $\quad f(x) = (x^1)^2 + \cdots + (x^n)^2$

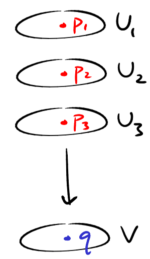
any $c \neq 0$ is regular value $\qquad f^{-1}(c) = \begin{cases} S^{n-1} & c > 0 \\ \emptyset & c < 0 \end{cases}$

Thus <u>S^{n-1} is a smooth mfd</u>. Much easier than building <u>charts</u> directly.

Thm M compact, $f: M \to N$ smooth, $\dim M = \dim N$,
$N_{reg,f} \subset N$ set of regular values for f:

$$\begin{aligned} \#: N_{reg,f} &\to \mathbb{Z}_{\geq 0} \\ q &\mapsto \# f^{-1}(q) \end{aligned}$$ is well defined, locally constant.

Pf If q is a regular value then $f^{-1}(q)$ is a 0-dimensional submanifold of M.
Thus it is a closed discrete subset of M. But M is compact. So $f^{-1}(q)$ is a finite set, say $f^{-1}(q) = \{p_1, \ldots, p_N\}$. Then, f is diffeo on nbhds U_i of p_i.



Put $V = \left[\bigcap_{i=1}^{N} \underbrace{f(U_i)}_{\text{open}}\right] \setminus \underbrace{f\left(M \setminus \bigcup_{i=1}^{N} U_i\right)}_{\text{compact}}$

Then V is open, $q \in V$, and $\# = N$ on V. ∎

Cor (Fundamental Thm of Algebra)
Say $f: \mathbb{C} \to \mathbb{C}$ non-constant polynomial. Then f has at least one root.

Pf Sketch First show f extends to a smooth map $S^2 \to S^2$. $f(\infty) = \infty$
Critical values of f are w s.t. $\exists z$ with $f(z) = w$, $f'(z) = 0$, plus perhaps ∞. There are only finitely many of these. (since $f'(c) = 0 \Rightarrow f'(z)$ is divisible by $(z-c)$)
So $S^2_{reg,f}$ connected, so by the Thm, $\#$ is constant on S^2_{reg}; and this constant can't be zero [else $f(S^2)$ is a finite set, which $\Rightarrow$ f is constant].
Thus if $0 \in S^2_{reg,f}$ $f^{-1}(0) \neq \emptyset$; and of course if $0 \in S^2_{crit,f}$ $f^{-1}(0) \neq \emptyset$ ∎

This used the fact that "most" values are regular values. This turns out to be very general:

Thm (Sard's Thm (weak version)) $f: M \to N$ smooth: $N_{reg,f}$ is dense in N.

Ex If $\dim M < \dim N$, $N \setminus f(M)$ is dense in N.
(space-filling curves can never be smooth)

Def 1) If $S = (a^1, b^1)\times(a^2,b^2)\times\cdots\times(a^n,b^n)\subset \mathbb{A}^n$ ("rectangle") then $\mu(S) = \prod_{i=1}^{n} (b^i - a^i)$.

2) $A\subset\mathbb{A}^n$ has <u>measure zero</u> if $\forall\varepsilon>0$, $\exists$ a countable cover $\{S_i\}$ of A by rectangles s.t. $\sum \mu(S_i) < \varepsilon$.

Prop 1) If A_i has measure zero $\forall i\in\mathbb{Z}_{>0}$ then $\bigcup_{i=1}^{\infty} A_i$ has measure zero.

2) $\mathbb{A}^k\subset\mathbb{A}^n$ has measure zero for $k<n$.

3) $U\subset\mathbb{A}^n$ open, $A\subset U$ measure zero, $f: U\to\mathbb{A}^n$ smooth $\Rightarrow$ $f(U)$ measure zero.

4) A rectangle does <u>not</u> have measure zero.

5) $A\subset\mathbb{A}^n$ closed, $\forall c\in\mathbb{R}$ $A\cap\{c\}\times\mathbb{A}^{n-1}$ has measure zero in $\mathbb{A}^{n-1}$ $\Rightarrow$ A has measure zero.
(Fubini)

Pf Sketch 1) Use $\sum_{i=1}^{\infty} \frac{\varepsilon}{2^i}=\varepsilon$.

2) Let $\{p_i\}$ be an enumeration of $\mathbb{Z}^k\subset\mathbb{A}^k$, then use rectangles centered on the p_i with all edge-lengths 1 except the n^{th}, n^{th} length $=\frac{\varepsilon}{2^i}$.

3) Cover A by balls $B_p\subset U$, such that $\overline{B}_p\subset U$; then on B_p, $\|df_p\|$ is bounded by some C_p. Topology on $\mathbb{R}^n$ has countable basis $\Rightarrow$ open covers have countable subcovers
(Pf: build subcover by: for each U_i in the basis, pick some V_α in the cover with $U_i\subset V_\alpha$, and otherwise pick nothing)

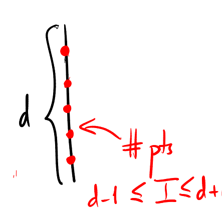
Thus can find $\{p_i\}$ countable, s.t. $\{B_{p_i}\}$ covers A.
$f(A)$ is countable union of $f(A\cap B_{p_i})$, so it's enough to show $f(A\cap B_{p_i})$ measure zero.

If $S\subset B_{p_i}$ cube of side length λ, $f(S)\subset S'$ cube of side length $C_{p_i}\sqrt{n}\,\lambda$.

Use this to show $f(A\cap B_{p_i})$ has measure zero.

4) We'll show if $S_1,S_2,\ldots$ rectangles covering $\overline{S}$ then $\sum \mathrm{vol}(S_i)\ge \mathrm{vol}(S)$.
Idea: say S has side lengths $d_1,\ldots,d_n$. Let $I(S)=\#$ integer points in S.
Then if all $d_i>1$,

$$\prod_i(d_i-1)\le I(S)\le\prod_i(d_i+1)$$



Since $\overline{S}$ is compact, $\exists$ <u>finite</u> collection $S_1,\ldots,S_N$ covering $\overline{S}$.
Call their side lengths $d_i(j)$ $j=1,\ldots,N$
Then $I(S)\le\sum I(S_j)$, so

$$\prod(d_i-1)\le I(S)\le\sum_j I(S_j)\le\sum_j\prod_i\left(d_i(j)+1\right)$$

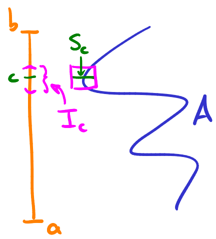
Now, rescale everything by $\lambda > 1$. Then $\lambda S_1, \ldots, \lambda S_N$ cover $\overline{\lambda S}$.

Get $\prod_i (\lambda d_i - 1) \leq \sum_j \prod_i (\lambda d_i(j) + 1)$

ie $\prod_i (d_i - \frac{1}{\lambda}) \leq \sum_j \prod_i (d_i(j) + \frac{1}{\lambda})$

and as $\lambda \to \infty$, this yields

$$\mu(S) = \prod_i d_i \leq \sum_j \prod_i d_i(j) = \sum_j \mu(S_j) \quad \text{as desired.}$$



5) $A \subset \mathbb{A}^n$ closed $\Rightarrow A$ is countable $\cup$ of compact sets $\Longrightarrow$ may reduce to case of A compact. Then $A \subset [a,b] \times \mathbb{A}^{n-1}$. Fix $c \in [a,b]$. Then let S_c be a union of (open) rectangles s.t. $A \cap (\{c\} \times \mathbb{A}^{n-1}) \subset \{c\} \times S_c$, with total volume $< \varepsilon$. $\exists$ some interval $I_c \subset [a,b]$ s.t. $A \cap (I_c \times \mathbb{A}^{n-1}) \subset I_c \times S_c$

[Pf otherwise, $\forall n$ $\exists p_n \in A \cap ([c - \frac{1}{n}, c + \frac{1}{n}] \times \mathbb{A}^{n-1})$, $p_n \notin \mathbb{R} \times S_c$, and passing to a subsequence convergent in A, ($\exists$ since A compact), we get limit $p \in \{c\} \times A$ but not in $\{c\} \times S_c$, ※]

Now given any covering of $[a,b]$ by open intervals, we can find a covering by subintervals with total length $< 2(b-a)$.

[Pf Take a countable subcovering by intervals (a_i, b_i). Then delete $\bigcup_{j=1}^{n-1} [a_j + \delta, b_j - \delta]$ from n^{th} interval, where $\delta = (b-a)/n \cdot 2^{n+1}$

This leaves a coll^n of open intervals, still covering $[a,b]$, total length $\leq (b-a) + \sum_n \frac{b-a}{n 2^{n+1}} n$]

Thus have a covering of A by rectangles of total volume $< 2(b-a)\varepsilon$. ∎

Def/Cor M a manifold: $S \subset M$ has measure zero if, $\forall$ charts (U, x) on M, $x(S \cap U)$ has measure zero.

If M is open subset of $\mathbb{A}^n$, this agrees with previous def.

Pf Use fact that smooth maps between affine spaces preserve measure zero ∎

Cor 1) If $S \subset M$ has measure zero, then $M \setminus S$ is dense.

2) If $f: M \to N$ and $\dim(M) < \dim(N)$ then $f(M)$ has measure zero.

Pf 1) $(\overline{M \setminus S})^c$ is open, but also $\subset S$, thus of measure zero, so can't contain any rectangle, so is empty.

2) Choose countable cover by charts on M to reduce to $f: U \to \mathbb{A}^n$ with $U \subset \mathbb{A}^m$, $m < n$. Then, take
$$F: U \times \mathbb{A}^{n-m} \to \mathbb{A}^n$$
$$(x, y) \mapsto f(x)$$

$U \times \{0\} \subset U \times \mathbb{A}^{n-m}$ has measure zero $\Longrightarrow F(U \times \{0\}) = f(U)$ does. ∎

Thm (Sard) $f: M \to N$ smooth, $C \subset M$ critical pts: $f(C) \subset N$ has measure zero.

Pf Choose countable cover of M, reduce to $f: U \to \mathbb{A}^n$ with $U \subset \mathbb{A}^m$. Induction on m.

Case $m = 0$: if $n = 0$ then $C = \emptyset$

$n > 0$ then $f(pt) = pt$ has measure zero.

Inductive step: $C \supset C_1 \supset C_2 \supset \cdots$ $\qquad C_i = \{$all partial derivatives of f of order $\leq i$ vanish$\}$

① $f(C \setminus C_1)$ measure zero.

② $f(C_i \setminus C_{i+1})$ measure zero for $i \geq 1$.

③ $f(C_k)$ measure zero for $k > \frac{m}{n} - 1$.

For ①: basic idea— straighten out one direction, then use inductive hypothesis.

Fix $x_0 \in C$. Say $\frac{\partial f^1}{\partial x^1}(x_0) \neq 0$. Define $h: U \longrightarrow \mathbb{A}^m$

$$x \longmapsto (f^1(x), x^2, \ldots, x^m)$$

this is local diffeo on nbhd U' of x_0, $h: U' \xrightarrow{\sim} V' \subset \mathbb{A}^m$

then $g = f \circ h^{-1}: V' \to \mathbb{A}^m$ has critical values $f(U' \cap C)$ $\qquad g(c, \ldots) = (c, \ldots)$ ("straightened" version of f)

and $g: V' \cap (\{c\} \times \mathbb{A}^{m-1}) \longrightarrow \{c\} \times \mathbb{A}^{n-1}$

so induces $g_c: V' \cap (\{c\} \times \mathbb{A}^{m-1}) \longrightarrow \mathbb{A}^{n-1}$

$$dg = \left(\begin{array}{c|c} 1 & 0 \\ \hline * & dg_c \end{array}\right)$$

this is surjective iff dg_c is surjective $\left[\text{Pf: use row operations to } \sim \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & dg_c \end{array}\right)\right]$

so crit pts of g on $V' \cap \{c\} \times \mathbb{A}^{m-1}$ are those of g_c

By induction, $\{$crit values of $g_c\}$ measure zero. $\{$Crit values of $g\}$ not nec closed, but $\{$crit pts of $g\}$ closed, and V' is an open subset of $\mathbb{A}^m$, thus a countable union of compact sets. Thus $\{$crit pts of $g\}$ is countable union of compact sets; thus $\{$crit values of $g\}$ is countable union of compact sets. Finally can apply Fubini to these to see that $\{$crit values of $g\}$ has measure zero.

So, x_0 has a nbhd U' s.t. $f(U' \cap C)$ has measure zero.

Cover $C \setminus C_1$ by countably many such nbhds $\Rightarrow f(C \setminus C_1)$ has measure zero.

② similar to ①

(3) We can cover U by countably many cubes. So, it's enough to show: $S \subset U$ cube of length $\delta \Rightarrow f(S \cap C_k)$ has measure zero.

Divide S into smaller cubes S', length δ/N. (N^m of them)

If $x \in S \cap C_k$ then $\exists$ a dep. on f, S only s.t. $\|f(x+h) - f(x)\| < a \cdot \|h\|^{k+1}$ (Taylor)

$\Rightarrow$ if S' contains some $x \in C_k$, $f(S')$ lies in cube with sides of length $2a\left[\sqrt{n}\,\frac{\delta}{N}\right]^{k+1}$

so this cube has volume $\sim N^{-(k+1)n}$

$\Rightarrow$ $f(S)$ lies in $\bigcup$ of cubes, total vol. $\sim N^{m-(k+1)n} \longrightarrow 0$ as $N \to \infty$.