Partitions of Unity

Def X top space:

1) a covering $\{V_\alpha\}_{\alpha\in A}$ is a refinement of $\{U_\alpha\}_{\alpha\in B}$ if $\exists\ \iota: B \hookrightarrow A$ and $V_{\iota(\alpha)} \subseteq U_\alpha \quad \forall \alpha \in B$.
   Not nec. $=$

2) a covering $\{U_\alpha\}$ of X is locally finite if $\forall p \in X\ \exists$ open nbhd W of p s.t. $W \cap U_\alpha = \phi$ for all but finitely many α.

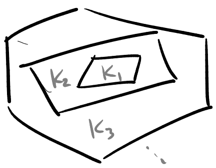
3) X is paracompact if every open cover has a locally finite refinement.

Ex S infinite discrete space, A uncountable $\Rightarrow \prod_A S$ is not paracompact.

Def M topological manifold: a regular open covering of M is a covering by charts (U_α, x_α) s.t. $x_\alpha(U_\alpha) \subset B_3(0)$, and the charts $(U_\alpha \cap x_\alpha^{-1}(B_1(0)), x_\alpha)$ still cover M.

Prop M topological manifold $\Rightarrow$ any open cover of M can be refined to a regular, locally finite open cover.
(in p^tc M is paracompact)

Pf



Idea: make covering locally finite by intersecting with compact "rings", then taking finite subcovers.

M second countable $\Rightarrow \exists$ countable open covering of M by charts. Each chart is countable $\cup$ of open sets with compact closure (just take all open sets of a countable basis which lie in the chart and have compact closure.) So, get charts U_i covering M, with $\overline{U_i}$ compact.
Then, let $K_1 = \overline{U_1}$ and $K_n = \overline{U_1 \cup \cdots \cup U_{p_n}}$ where p_n is 1st int s.t. $K_{n-1} \subset U_1 \cup \cdots \cup U_{p_n}$.
Thus $K_n \subset K_{n+1}^\circ$, each K_n compact, $\bigcup K_n = M$.

Now, take some open cover $\{W_\alpha\}$. If $p \in W_\alpha$ then $p \in K_{n+1} \setminus K_n^\circ$ for some n.
Then take a chart $(U_{p,\alpha}, \phi_{p,\alpha})$ with $U_{p,\alpha} \subset K_{n+2} \setminus K_{n-1}^\circ \cap V_\alpha$, $\phi_{p,\alpha}(U_{p,\alpha}) \subset B_0(3)$, $\phi_{p,\alpha}(p) = 0$, set $V_{p,\alpha} = \phi_{p,\alpha}^{-1}(B_0(1))$.

Now fix n. The $V_{p,\alpha}$ for $p \in K_{n+1} \setminus K_n^\circ$ cover $K_{n+1} \setminus K_n^\circ$, without leaving $K_{n+2} \setminus K_{n-1}^\circ$.
Pass to finite subcover: get a set $\{V_{i,n}\}$ of charts. Then take $\bigcup$ over all n to get the desired regular, locally finite subcover of $\{W_\alpha\}$. ∎

Def Given an open cover $\{U_\alpha\}$ of X a partition of unity subordinate to $\{U_\alpha\}$ is a collection of functions $\phi_i : X \to \mathbb{R}$, s.t.

1) $0 \le \phi_i(x) \le 1 \quad \forall x \in X$
2) $\forall x \in X$, $\exists$ nbhd W of x s.t. all but finitely many $\phi_i = 0$ on W.
3) each $\phi_i = 0$ except on a closed subset of one U_α.
4) $\forall x \in X$, $\sum_i \phi_i(x) = 1$.

Prop X paracompact, $\{U_\alpha\}$ open cover $\Rightarrow$ partitions of unity subord. to $\{U_\alpha\}$ exist.
(We won't use, so don't prove.)

Prop M smooth mfd, $\{U_\alpha\}$ open cover $\Rightarrow$ smooth partitions of unity subord. to $\{U_\alpha\}$ exist.

Pf Refine $\{U_\alpha\}$ to a regular, locally finite covering by charts (U_α, x_α).

$\exists$ smooth function $\varphi: \mathbb{A}^n \to \mathbb{R}$ with $\varphi = 1$ on $B_0(1)$, $\varphi = 0$ outside $B_0(2)$. [HW]

Then, get smooth function $f_\alpha: M \to \mathbb{R}$ by $f_\alpha(p) = \begin{cases} \varphi(x_\alpha(p)) & p \in U_\alpha \\ 0 & p \notin U_\alpha \end{cases}$

and define $\phi_\alpha = \dfrac{f_\alpha}{\sum_\alpha f_\alpha}$ $\longleftarrow$ [finite sum at any $p \in M$]

Check it has the desired properties 1-4 ∎

Whitney Embedding

Lemma M compact mfd: $\exists n \in \mathbb{N}$ s.t. $\exists\, \iota: M \hookrightarrow \mathbb{A}^n$ embedding.

Pf Cover M by finitely many charts (U_i, x_i). $i = 1, \ldots, k$

Take a part^n of unity ϕ_i subordinate to the covering U_i.

Then take $n = k \cdot (\dim M) + k$

and define $\iota = (\tilde{x}_1, \ldots, \tilde{x}_k, f_1, \ldots, f_k)$ where $\tilde{x}_i(p) = \begin{cases} f_i x_i(p) & p \in U_i \\ 0 & p \notin U_i \end{cases}$

If $f_i(p) = f_i(p')$ $\forall i$ then p, p' both lie in some common chart U_i, but then $\tilde{x}_i(p) = \tilde{x}_i(p')$ shows $x_i(p) = x_i(p')$, so $p = p'$! So ι is injective.

And $d\iota(\xi) = (x_1 df_1(\xi) + f_1 dx_1(\xi), \ldots, df_1(\xi), \ldots)$

thus if $d\iota(\xi) = 0$ then $dx_i(\xi) = 0$ for at least one i

but this is impossible since x_i is diffeomorphism! So ι is immersion.

Thus ι is embedding, (since M is compact, so ϕ proper) ∎

Thm M compact mfd: $\exists\, \iota: M \hookrightarrow \mathbb{A}^{2(\dim M)+1}$ embedding.

Pf We may assume $M \subset \mathbb{R}^n$. Also suppose $n > 2m+1$. ($m = \dim M$)

Then, we will produce a linear projection $\pi: \mathbb{R}^n \to \mathbb{R}^{n-1}$

such that $\pi|_M$ is embedding. By induction this gives the desired result.

Define $h: M \times M \times \mathbb{R} \longrightarrow \mathbb{R}^n$ $\qquad h(p,p',t) = t(p-p')$

$\qquad g: TM \longrightarrow \mathbb{R}^n$ $\qquad g(p,\xi) = \xi$ $\qquad$ (or more precisely $d\iota_p(\xi)$ $\quad \iota: M \hookrightarrow \mathbb{A}^n$)

Since $n > 2m+1$, Sard $\Rightarrow$ $\exists$ some $a \in \mathbb{R}^n$ not in image of g or h. Note $a \neq 0$.

We'll show that the projection $\pi: \mathbb{R}^n \to a^\perp \simeq \mathbb{R}^{n-1}$ works.

First, injectivity: if $\pi(p) = \pi(p')$, $p,p' \in M$, then $p - p' = \lambda a$,

$\Rightarrow$ a is in the image of h or $\lambda = 0$.

Second, immersivity: if $d\pi_p(\xi) = 0$ for some $(p,\xi) \in TM$, then $\xi = ta$ for some t

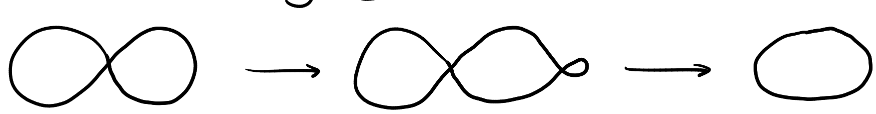
$\Rightarrow$ a is in the image of g or $t = 0$.

And M compact, so $\pi|_M$ proper. Thus $\pi|_M$ is embedding. $\blacksquare$

Thm M compact mfd: $\exists \iota: M \hookrightarrow \mathbb{A}^{2(\dim M)}$ embedding.

__Pf Sketch__ This is harder. The above pf shows $\exists$ an __immersion__ but not an __injective__ one!

Need a clever trick for getting rid of self-intersections ("Whitney trick")

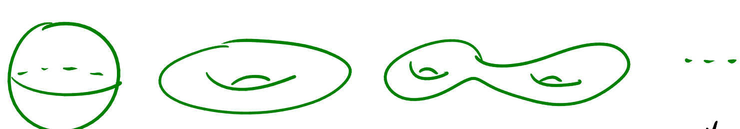


__Rk__ 1) The bound $2(\dim M)$ is not sharp, except in case $\dim M = 2^k$.

2) Classification of compact connected 1-manifolds: S^1 is the only one.

Indeed, S^1 embeds in $\mathbb{A}^2$.

3) Compact connected __orientable__ 2-manifolds all embed in $\mathbb{A}^3$:



But __non-orientable__ 2-manifolds embed only in $\mathbb{A}^4$.

4) Whitney embedding also works for __non-compact__ mfds, but the pf is much more technical.