**Definition:** Let $X$ be a topological space.

1. A covering $\left\{ U_\alpha \right\}_{\alpha \in A}$ is a refinement of $\left\{ U_\alpha \right\}_{\alpha \in B}$ if $\exists \alpha : B \to A$ and $V_\alpha = \bigcup_{V_\alpha} U_\alpha$ for all $\alpha \in B$.

2. A covering $\left\{ U_\alpha \right\}$ of $X$ is locally finite if $\forall p \in X$ there exists an open set $W$ s.t. $W \cap U_\alpha = \emptyset$ for all but finitely many $\alpha$.

3. $X$ is paracompact if every open cover has a locally finite refinement.

**Example:** If $S$ is an infinite discrete space, then $S$ is not paracompact.

**Definition:** A topological manifold $M$ is a covering by charts $(U_\alpha, x_\alpha)$ s.t. $x_\alpha(U_\alpha) \subseteq B_3(0)$, and the charts $(U_\alpha \cap x_\alpha^{-1}(B_3(0)), x_\alpha)$ still cover $M$.

**Proposition:** Any open cover of $M$ can be refined to a regular, locally finite open cover in $p^\ast M$ (the base change).

**Proof:** For a second-countable manifold $M$, each chart is countable $\Rightarrow$ countable open covering of $M$ by charts. Each chart $U$ of open sets with compact closure (just take all open sets of a countable basis which lie in the chart and have compact closure.)

Thus, $K_n = \bigcup_{i=1}^n U_i$ is compact. Then, let $K = \bigcup_{i=1}^n U_i$ and $K_n = \bigcup_{i=1}^n U_i \cup \cdots \cup U_n$ where $p_n$ is the $n$th int. set. $K_n \subseteq U_1 \cup \cdots \cup U_n$.

Now, take an open cover $\left\{ W_\alpha \right\}$. If $p \in W_\alpha$, then $p \in K_{n+1} \setminus K_n$ for some $n$.

Then, take a chart $(U_\alpha, x_\alpha)$ with $U_\alpha \subseteq K_{n+1} \setminus K_{n-1}$ and $x_\alpha(U_\alpha) \subseteq B_3(0)$, $\phi_{p_n}(p) = 0$.

Set $V_{p_n} = \phi_{p_n}^{-1}(B_3(0))$.

Now, fix $n$. The $V_{p_n}$ for $p \in K_{n+1} \setminus K_n$ cover $K_{n+1} \setminus K_n$ without leaving $K_{n+2} \setminus K_{n-1}$.

Pass to finite subcover: get a set $\left\{ V_{p_n} \right\}$ of charts. Then take $U$ over all $n$ to get the desired regular, locally finite subcover of $\left\{ W_\alpha \right\}$.

**Definition:** Given an open cover $\left\{ U_\alpha \right\}$ of $X$ a partition of unity subordinate to $\left\{ U_\alpha \right\}$ is a collection of functions $\phi_i : X \to \mathbb{R}$ s.t.

1. $0 \leq \phi_i(x) \leq 1 \quad \forall x \in X$.
2. $\forall x \in X, \exists \text{ a } U \text{ of } x \text{ s.t. all but finitely many } \phi_i = 0 \text{ on } U.$
3. Each $\phi_i = 0$ except on a closed subset of one $U_\alpha$.
4. $\forall x \in X, \sum \phi_i(x) = 1$.

**Proposition:** If $X$ is paracompact, then $\left\{ U_\alpha \right\}$ open cover $\Rightarrow$ partition of unity subordinate to $\left\{ U_\alpha \right\}$ exists.

(We won't use, so don't prove.)
Prop M smooth mfd, \{U_\alpha\} open cover \implies smooth partition of unity subord to \{U_\alpha\} exist.

Pf Define \{U_\alpha\} to a regular, locally finite covering by charts \((U_\alpha, x_\alpha)\).

\exists \text{ smooth function } \varphi: \mathbb{R}^n \to \mathbb{R} \text{ with } \varphi=1 \text{ on } B_0(1), \varphi=0 \text{ outside } B_0(2). [HW]

Then, get smooth function \(f_\alpha: M \to \mathbb{R}\) by \(f_\alpha(p) = \begin{cases} 
\varphi(x_\alpha(p)) & p \in U_\alpha \\
0 & p \notin U_\alpha
\end{cases}
\)

and define \(\phi_\alpha = \frac{f_\alpha}{\sum_\alpha f_\alpha} \left[\text{finite sum at any } p \in M\right]
\)

Check it has the desired properties 1-4.

Whitney Embedding

Lemma M compact mfd: \exists n \in \mathbb{N} \text{ s.t. } \exists \iota: M \to \mathbb{A}^n \text{ embedding.}

Pf Cover \(M\) by finitely many charts \((U_i, x_i)\). \(i=1,\ldots,k\)

Take a part of unity \(\phi_i\) subord to the covering \(U_i\).

Then take \(n = k \cdot (\dim M) + k\)

and define \(\iota = (x_1, \ldots, x_k, f_1, \ldots, f_k)\) where \(x_i(p) = \begin{cases} f_i x_i(p) & p \in U_i \\
0 & p \notin U_i
\end{cases}\)

If \(f_i(p) = f_i(p') \forall i\) then \(p, p'\) both lie in some common chart \(U_i\), but then \(x_i(p) = x_i(p')\) shows \(x_i(p) = x_i(p')\) so \(p = p'\). So \(\iota\) is injective.

And \(d\iota(\mathfrak{x}) = (x_1 df_1(\mathfrak{x}) + f_1 dx_1(\mathfrak{x}), \ldots, df_k(\mathfrak{x}))\)

Thus if \(d\iota(\mathfrak{x}) = 0\) then \(dx_i(\mathfrak{x}) = 0\) for at least one \(i\)

but this is impossible since \(x_i\) is diffeomorphism! So \(\iota\) is immersion.

Thus \(\iota\) is embedding, (since \(M\) is compact, so \(\phi\) proper)

Thm M compact mfd: \exists \iota: M \to \mathbb{A}^{2(\dim M) + 1} \text{ embedding.}

Pf We may assume \(M \subseteq \mathbb{R}^n\). Also suppose \(n \geq 2m+1\). \((m = \dim M)\)

Then we will produce a linear projection \(\pi: \mathbb{R}^n \to \mathbb{R}^{n-1}\)
such that $\pi^*_M$ is embedding. By induction this gives the desired result.

Define $h : MX \times \mathbb{R} \to \mathbb{R}^n$ 
$h(p, p', t) = t(p - p')$

$g : TM \to \mathbb{R}^n$ 
$g(p, \xi) = \xi$  (or more precisely $d\pi^*_p(\xi) : M \to \mathbb{R}^n$)

Since $n > 2m + 1$, $S^m \to \mathbb{R}^n$ is not in the image of $g = h$. Note $a \neq 0$.

We'll show that the projection $\pi : Tr^n \to a^t \cong \mathbb{R}^{n-1}$ works.

First, injectivity: if $\pi(p) = \pi(p')$, $pp' \in M$, then $p - p' = \lambda a$, 
$\implies a$ is in the image of $h$ or $\lambda = 0$.

Second, immersivity: if $d\pi^*_p(\xi) = 0$ for some $(p, \xi) \in TM$, then $\xi = ta$ for some $t$
$\implies a$ is in the image of $g$ or $t = 0$.

And $M$ compact, so $\pi|_M$ proper. Thus $\pi|_M$ is embedding.

**Theorem:** Compact manifold $M \to \mathbb{A}^{2(\dim M)}$ embeds.

**Proof Sketch:** This is harder. The above proof shows $\exists$ an immersion but not an injective one!

Need a clever trick for getting rid of self-intersections ("Whitney trick")

\[
\infty \quad \rightarrow \quad \infty \quad \rightarrow \quad \circ
\]

**Remarks:**

1) The bound $2(\dim M)$ is not sharp, except in case $\dim M = 2^k$.

2) Classification of compact connected 1-manifolds: $S^1$ is the only one.

   Indeed, $S^1$ embeds in $\mathbb{A}^2$.

3) Compact connected orientable 2-manifolds all embed in $\mathbb{A}^3$:

   \[
   \cdots \quad \circ \quad \circ \quad \circ \quad \cdots
   \]

   But non-orientable 2-manifolds embed only in $\mathbb{A}^4$.

4) Whitney embedding also works for non-compact manifolds, but the proof is much more technical.