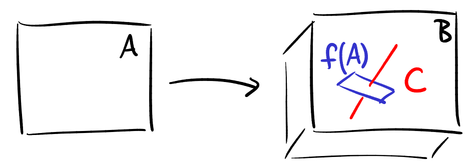
Transversality

We've seen that if $q \in N$ is <u>regular value</u> for $f: M \to N$
then $f^{-1}(q)$ is <u>submanifold</u> of M.

Key generalization: suppose $Q \subset N$. What can we say about $f^{-1}(Q)$?

<u>Def</u> $f: M \to N$ smooth, $Q \subset N$ submfd:
say f is <u>transverse to Q</u>, $f \pitchfork Q$, if $\forall p \in f^{-1}(Q)$, $\operatorname{im}(df_p) + T_{f(p)}Q = T_{f(p)}N$.

<u>Ex</u>
1) If $Q = \{q\}$ then $f \pitchfork Q \iff q$ is regular value of f.
2) If $Q = N$ then $f \pitchfork Q$ always.
3) If A, B affine spaces, $C \subset B$ affine subspace, $f: A \to B$ affine map
V, W assoc vector sp, $Y \subset W$,

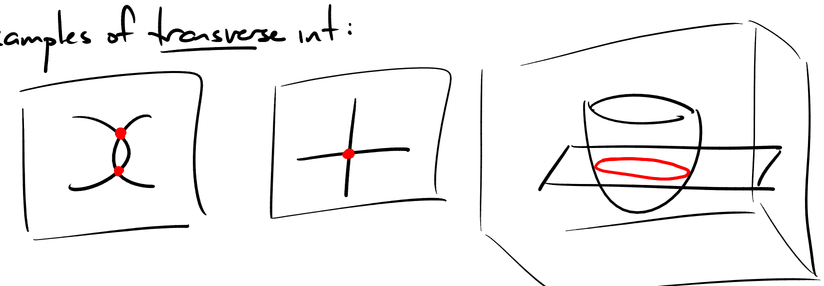


then $f \pitchfork C \iff$ associated <u>linear</u> map $df: V \to W \to W/Y$ surjective.

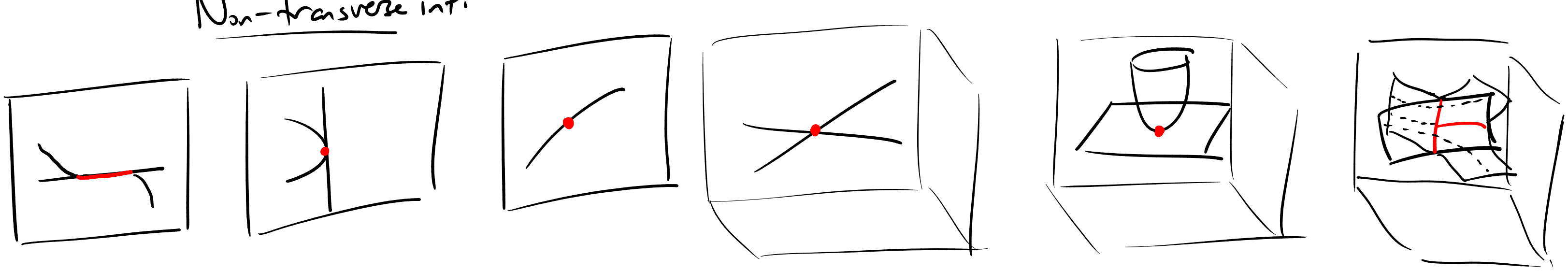
4) If $\dim M + \dim Q < \dim N$
then $f \pitchfork Q \iff f^{-1}(Q) = \emptyset$ ie $f(M) \cap Q = \emptyset$.

5) If f is <u>inclusion</u> $M \hookrightarrow N$ then write $M \pitchfork Q$ for $f \pitchfork Q$.
Symmetric between M, Q: $T_pM + T_pQ = T_pN$ for $p \in M \cap Q$.

Examples of <u>transverse</u> int:



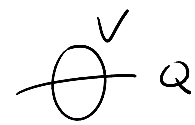
<u>Non-transverse</u> int:



Thm If $f: M \to N$ smooth, $Q \subset N$ submfd, $f \pitchfork Q$, then $f^{-1}(Q) \subset M$ submfd.

Pf Fix $p \in f^{-1}(Q)$, $q = f(p)$, chart (V, y) on N around q such that $y(Q) = \{x^1 = \dots = x^k = 0\} \subset y(V)$.

Consider
$$\begin{tikzcd} f^{-1}(V) \arrow[r, "f"] \arrow[d, phantom, sloped, "\subset"] & V \arrow[r, "y"] \arrow[d, phantom, sloped, "\subset"] & \mathbb{A}^n \arrow[r, "\pi"] & \mathbb{A}^n / \mathbb{A}^k \\ M & N & & \end{tikzcd}$$



$f \pitchfork Q \Rightarrow \operatorname{im} d(y \circ f) \oplus \mathbb{A}^k = \mathbb{A}^n \Rightarrow d(\pi \circ y \circ f)$ surjective. $f^{-1}(Q) \cap f^{-1}(V) = (\pi \circ y \circ f)^{-1}(0)$

Thus $f^{-1}(Q) \cap f^{-1}(V)$ is a submanifold of M. In particular, this says $f^{-1}(Q)$ is covered by open sets U s.t. $f^{-1}(Q) \cap U$ is submanifold of M. This $\Rightarrow f^{-1}(Q)$ is submanifold ∎

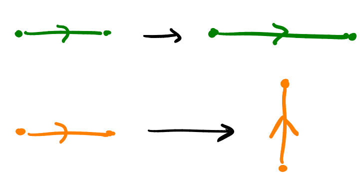
Cor If $M \pitchfork Q$ then $M \cap Q$ is a submanifold of M (or of Q by symmetry, or of N)

Homotopy

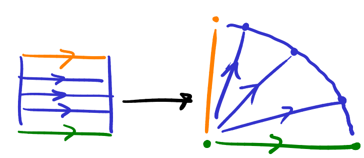
Def If $f_0, f_1 : M \to N$ smooth, $f: M \times [0,1] \to N$ smooth, $f(p,0) = f_0(p)$, $f(p,1) = f_1(p)$ then f is a smooth homotopy between f_0 and f_1, and we say f_0 is smoothly homotopic to f_1.
Also write $f_t : M \to N$ $\quad f_t(p) = f(p,t)$.

Ex $f_0 : I \to \mathbb{A}^2 \qquad f_0(s) = (0, s)$

$f_1 : I \to \mathbb{A}^2 \qquad f_1(s) = (s, 0)$



are smoothly homotopic through $f_t : I \to \mathbb{A}^1$, $f_t(s) = (s \sin(\frac{\pi}{2} t), s \cos(\frac{\pi}{2} t))$



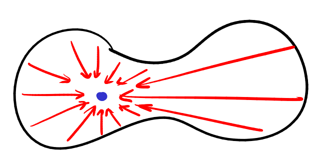
A basic question in topology: study homotopy classes of maps.

Prop Any smooth map $f_0 : M \to S^n$ with $\dim(M) < n$ is smoothly homotopic to a constant map.

Pf Sard $\Rightarrow \exists q \in S^n$ with $f_0^{-1}(q) = \phi$.

By stereographic projection, have a chart (U, x) with $U = S^n \setminus q$. $\quad x : U \hookrightarrow \mathbb{R}^n$

Define $f : M \times [0,1] \to S^n$
$$(p, t) \longmapsto x^{-1}[(1-t)\, x(p)]$$



∎

Cor S^n is simply connected for $n>1$.

Related: what properties of a map are preserved under a small homotopy?

Given a property P, call P stable if, $\forall$ maps $f_0: M \to N$ with property P, and homotopies f_t of f_0, $\exists\, \varepsilon > 0$ s.t. $f_t: M \to N$ has property P $\forall\, t < \varepsilon$.

Thm If M is compact, then the following properties of smooth maps $f: M \to N$ are stable:

1) local diffeo,
2) immersion,
3) submersion,
4) transversal to a fixed closed submfd $Q \subset N$,
5) embedding,
6) diffeo.

Pf 2) f_0 immersion, f_t homotopy.

M compact $\Rightarrow$ enough to show $\forall\, p_0 \in M$ $\exists$ nbhd $U \subset M \times [0,1]$ of $(p_0, 0)$ s.t. f_t is immersion at p $\forall\, (p,t) \in U$.

Using charts, reduce to $M \subset \mathbb{A}^m$ open, $N \subset \mathbb{A}^n$ open.

Then immersivity at $p_0 \Rightarrow$ some $m \times m$ minor of the Jacobian is $\neq 0$
$\Rightarrow$ this minor still $\neq 0$ in a small nbhd U.

2) $\Rightarrow$ 1) just by taking special case $m = n$.

3) similar, just use $n \times n$ minors instead.

4) also similar to 3)

5) $\cdots$

$\blacksquare$