Prop M smooth mfd, $f: M \to \mathbb{R}$, $c \in \mathbb{R}$ regular value for f:

$f^{-1}((-\infty, c])$ is a manifold with boundary $f^{-1}(c)$.

Ex $M =$ torus in $\mathbb{A}^3$
$f =$ "height function"

Ex $M = \mathbb{R}^n$ $f(x) = \|x\|^2$ $\to$ closed ball is manifold whose boundary is S^{n-1}

Pf $f^{-1}((-\infty, c))$ open $\subset M$, so for any $p \in f^{-1}((-\infty, c))$ just restrict charts from M.

For $p \in f^{-1}(c)$ build a chart of form $(f-c, x^2, \dots, x^n)$.

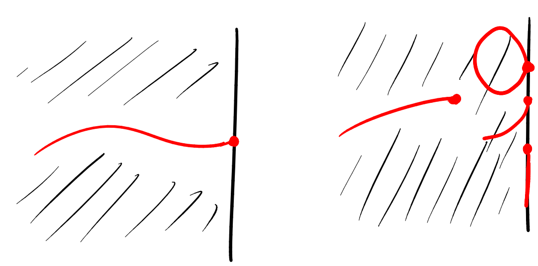
(Why is this possible? The map $f: M \to \mathbb{R}$ is a submersion at $f^{-1}(c)$, so we can always find a chart in which f looks like $(x^1, \dots, x^n) \mapsto x^1$) ∎

Thm (Sard) M manifold with boundary, N manifold, $f: M \to N$ smooth:
set of simultaneous regular vals. for f and $\partial f: \partial M \to N$ is dense in N.

Pf A regular point for $\partial f: \partial M \to N$ is automatically regular point for $f: M \to N$.
Take the simultaneous regular vals for $\partial f: \partial M \to N$ and $f|_{\mathrm{Int}(M)}: \mathrm{Int}(M) \to N$.
Ordinary Sard applies to each of these. ∎

Def M mfd with boundary: $N \subset M$ is <u>neat submanifold</u> if $\forall p \in N$ $\exists$ chart (U, x) of M s.t. $x(N \cap U) = (\mathbb{A}^n)^- \cap x(U) \subset (\mathbb{A}^m)^-$

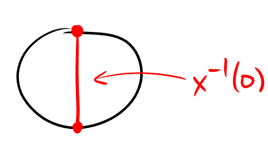
(so $N \cap \partial M = \partial N$ and $N \pitchfork \partial M$)



Prop M mfd with boundary, N mfd, $f: M \to N$ smooth, $q \in N$ regular value for f and ∂f:
then $f^{-1}(q) \subset M$ is <u>neat</u> submanifold of M.

(thus $\dim M > \dim N$ if $\partial M \neq \emptyset$)

Ex $B = \{x^2 + y^2 \leq 1\} \subset \mathbb{A}^2$ $\quad$ $B \to \mathbb{R}$, $(x,y) \mapsto x$



Pf For $p \in f^{-1}(q) \cap \partial M$, choose boundary chart (U, x) around p, and chart (V, y) around q, with $y(q) = 0$. Then q regular for $\partial f \Rightarrow \left(\frac{\partial \tilde{y}^i}{\partial x^j}\right)_{\substack{i=1,\dots,n \\ j=2,\dots,m}}$ has rank n $(\tilde{y} = y \circ f)$

After re-ordering the x^j, can arrange that $\left(\frac{\partial \tilde{y}^i}{\partial x^j}\right)_{\substack{i=1,\dots,n \\ j=m-n+1,\dots,m}}$ is invertible.

Then consider the map $U \to \mathbb{A}^n$

$$p \mapsto \hat{x}(p) = (x^1(p), \dots, x^{m-n}(p), \tilde{y}^{m-n+1}(p), \dots \tilde{y}^m(p))$$

This has differential $\left[\begin{array}{c|c} 1 & * \\ \hline 0 & \text{invertible} \end{array}\right]$ so it is local diffeomorphism on some $\hat{U} \subset U$.

Then $\hat{x}(f^{-1}(q)) = (\mathbb{A}^{m-n})^- \cap \hat{x}(\hat{U}) \subset (\mathbb{A}^m)^-$. ∎

More generally:

Prop M mfd with boundary, N mfd, $Q \subset N$ submfd, $f: M \to N$ transversal to Q: $f^{-1}(Q)$ is <u>neat</u> submanifold of M.

Pf Sketch Reduce locally to case $Q = pt$, as we did for the case <u>without</u> boundary.