Def M smooth mfd: 1) a <u>flow on M</u> is a 1-param family of diffeos $\varphi_t : M \to M$ $\quad t \in \mathbb{R}$
with $\varphi_t \circ \varphi_s = \varphi_{t+s}$, $\varphi_0 = 1$, s.t. $\varphi : M \times \mathbb{R} \to M$ is smooth.

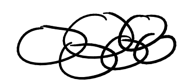
2) <u>the velocity field of φ</u> is the vector field $X \in \Gamma(TM)$
given by $X(p) = \frac{d}{dt}\Big|_{t=0} \varphi_t(p)$ (precisely: $d\varphi_{p,0}(\frac{\partial}{\partial t})$)

Ex If $M = \mathbb{A}^1$ and $X = \frac{\partial}{\partial x}$, then $\varphi_t(x) = x+t$ is a flow with velocity field X.

Prop Given any compactly supported vector field $X \in \Gamma(TM)$ there exists a flow
$\varphi_t : M \to M$ whose velocity field is X.

Rk Compactness is important here, e.g. take $M = (0,1)$ and $X = \frac{\partial}{\partial x}$.

Pf See Spivak vol. 1, chapter 5, Thm 6, p. 206.
Main idea: use <u>local</u> existence of flows on $\mathbb{A}^m$ for short time ε
cover M by finite # balls to get <u>global</u> existence
for time $\varepsilon = \min(\varepsilon_i)$



Then define the flow for larger times by <u>iterating</u> e.g. $\varphi_{3/2\,\varepsilon} = \varphi_{3/4\,\varepsilon} \circ \varphi_{\frac{3}{4}\varepsilon}$ ∎

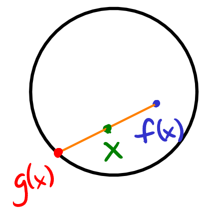
Lemma If N connected, $q, q' \in N$ then $\exists\, \varphi : [0,1] \times N \to N$
with $\varphi_0 = id_N$, φ_t diffeo, $\varphi_1(q) = q'$.

Pf Sketch Take embedded path q_t from q to q'. $dq_t(\frac{\partial}{\partial t})$ is a vector field <u>on this path</u> inside N.
Extend it to a compactly supported vector field $X \in \Gamma(TN)$, then let φ_t be the flow of X.

Thm (Brouwer) $B \subset \mathbb{R}^n$ closed unit ball, $f: B \to B$ smooth: f has a fixed point.

Pf Suppose $\exists f: B \to B$ smooth w/o fixed point.

Then construct $g: B \to \partial B$ by



Note if $x \in \partial B$ then $g(x) = x$.

And, g is smooth: indeed $g(x) = x + \hat{t}(x)\,(x - f(x))$ so all we need is to see that $\hat{t}(x)$ depends smoothly on x, and this follows from implicit function thm (from HW):

$$F: B \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(x,t) \longmapsto \|x + t(x - f(x))\|^2$$

$$\frac{\partial F}{\partial t} = 2\big(x - f(x)\big)\cdot\big(x + t(x - f(x))\big)$$
$$\neq 0 \quad \text{when } F(x,t) = 1$$

[since $F(x,t) = \|x\|^2 + 2t\,(x - f(x))\cdot x + t^2\|x - f(x)\|^2$

ie $\qquad 1 = \|x\|^2 + 2t\,(x - f(x))\cdot(x + t(x - f(x))) - t^2\|x - f(x)\|^2$

so either the cross term is >0 or $\|x\|^2 = 1$, $t = 0$; but in that case $(x - f(x))\cdot x$ is still >0 so long as $f(x) \neq x$]

$\Rightarrow \exists\ \hat{t}(x)$ smooth, s.t. $F(x, \hat{t}(x)) = 1$. (Can also get this $\hat{t}$ explicitly by solving a quadratic eq^n)

Thus, g gives a retraction from B to ∂B. $\times\!\!\!\times$ ∎

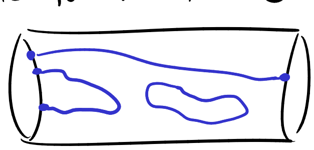
Def/Prop M compact, N connected, $\dim M = \dim N$, $f: M \to N$ smooth:

$\deg_2 f = \# f^{-1}(q) \bmod 2$, $q \in N$ regular value.

This is independent of q and also invariant under smooth homotopy of f.

Pf ① For homotopy: Suppose $F: [0,1] \times M \longrightarrow N$, and q regular for ∂F.

If also q is regular for F then $S = F^{-1}(q)$ is a neat submanifold-with-boundary, of dimension 1



so $\# f_0^{-1}(q) = \# f_1^{-1}(q) \bmod 2$.

If q is not regular for F, perturb it slightly so that it becomes regular for F: $\exists$ nbhds U_0, U_1 of q, where $\# f_0^{-1}, \# f_1^{-1}$ are constant resp. (we proved earlier these $\exists$), and

can find some $q' \in U_0 \cap U_1$ which is regular value for F (Sard), then
$\# f_0^{-1}(q) = \# f_0^{-1}(q') = \# f_1^{-1}(q') = \# f_1^{-1}(q)$.

(2) For independence of q: need the Lemma above, giving a family of diffeos
$\varphi_t : N \to N$, $\varphi_0(q) = q$, $\varphi_1(q) = q'$.

Then, given q, q' regular for f, take φ_t as above, consider $F : [0,1] \times M \to N$
$$(t, p) \mapsto \varphi_t(f(p))$$

q is a regular value for ∂F (since both q, q' were regular for f)
Then apply homotopy invariance from (1) above. ∎

Cor M compact smooth mfd $\Rightarrow$ the identity map $f : M \to M$ is not smoothly homotopic to a constant map.

Pf The identity has $\deg_2 f = 1 \pmod 2$ while a constant map has $\deg_2 f = 0 \pmod 2$. ∎