Def 1) An **algebra** is a vector space $A$ with an associative product $A \times A \to A$ and unit $1 \in A$.

2) An alg. $A$ is **$\mathbb{Z}$-graded** if $A = \bigoplus_{n=-\infty} A_n$ and $A_n \cdot A_m \subset A_{n+m}$

Elements $x \in A_n$ are called *homogeneous* of degree $n$. Write this $|x| = n$.

3) A $\mathbb{Z}$-graded alg $A$ is **(graded) commutative** if $x \cdot y = (-1)^{|x||y|} y \cdot x$

for $x, y$ homogeneous.

**Rk** $1 \in A_0$.

**Def** $U$ vector space: the **tensor algebra** $T(V)$ is $\mathbb{Z}$-graded alg $T(V) = \bigoplus_{n=0} \otimes^n$ $\infty$

$$= \mathbb{R} \oplus V \oplus V \otimes V \oplus \cdots$$

with product $V^\otimes m \times V^\otimes n \to V^\otimes m+n$ given by the standard bilinear map (recall for any vector space $V$, $W \ni V \times W \to V \otimes W$ bilinear).

**Prop** (Universal property of $T(V)$) If $A$ is any algebra and $j : V \to A$ any linear maps, then there is algebra hom $\tilde{j} : T(V) \to A$ s.t. the diagram

$\begin{array}{ccc}
T(V) & \xrightarrow{\tilde{j}} & A \\
V & \xrightarrow{j} & A \\
\end{array}$

commutes, where $i : V \hookrightarrow T(V)$ is the obvious inclusion.

**Pf** Uniqueness:
Commutativity forces $\tilde{j}(x) = j(x)$ for $x \in V$. These elements generate $T(V)$. Thus algebra hom property determines $\tilde{j}$ on the rest of $T(V)$.

Existence:

$\begin{array}{ccc}
\tilde{j}_n : V^n & \to & A \\
(x_1, \ldots, x_n) & \mapsto & \tilde{j}(x_1) \cdots \tilde{j}(x_n) \\
\end{array}$

is multilinear, so factor through $V^n \xrightarrow{j_n} A$

then define $\tilde{j}$ to be the linear map agreeing with $\tilde{j}_n$ on $V^\otimes n \subset T(V)$,

check $\tilde{j}(x_1 \otimes \cdots \otimes x_n) \tilde{j}(y_1 \otimes \cdots \otimes y_m) = j(x_1) \cdots j(x_n) j(y_1) \cdots j(y_m) = \tilde{j}(x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m)$
This universal property determines $T(V)$ up to unique $\cong$:
if there another $T'(V)$ with the same property, then get the diagram

\[
\begin{array}{c}
V & \xrightarrow{T} & T(V) \\
\downarrow & & \downarrow \varphi \\
\downarrow & & \downarrow \varphi' \\
T(V) & \xrightarrow{\varphi'} & T'(V)
\end{array}
\]

and the uniqueness part of the universal property shows $\varphi' \circ \varphi = 1$, so $\varphi$ is $\cong$.

If $\{e_i\}$ basis for $V$, $\dim V = k$
then $\{e_1 \otimes \cdots \otimes e_n\}$ basis for $V^\otimes n$, $\dim V^\otimes n = k^n$

**Def** Vector space: **exterior algebra** $\Lambda(V)$ is $T(V)/I$ where $I$ is 2-sided ideal generated by $\{v \otimes v : v \in V\}$.

**Prop** (Universal property of $\Lambda(V)$) If $A$ is any algebra and $j: V \to A$ any linear map
s.t. $j(x)^2 = 0 \ \forall x \in V$, then algebra $\tilde{j}: \Lambda(V) \to A$ s.t.

\[
\Lambda(V) \xrightarrow{\tilde{j}} A
\]

commutes.

**Pf** We have $\tilde{j}: T(V) \to A$ by universal property, and it descends to $\Lambda(V)$
since $\tilde{j}(x^2) = \tilde{j}(x)^2 = j(x)^2 = 0$. Uniqueness also follows from the uniqueness
for $T(V) \to A$.

Notation: Write the product in $\Lambda(V)$ as $\wedge$.

**Prop** $\Lambda(V)$ is $\mathbb{Z}$-graded and (graded) commutative.

**Pf** $\mathbb{Z}$-grading follows from fact that $I$ is generated by homogeneous elements. For commutativity:
First note for $x, y \in V$ we have $(x+y) \wedge (x+y) = 0$, so $(x \wedge y) + (y \wedge x) + (x \wedge y) + (y \wedge x) = 0$, so $x \wedge y = -y \wedge x$.
Enough to check on decomposable elements:

\[
(x_1 \wedge \cdots \wedge x_n) \wedge (y_1 \wedge \cdots \wedge y_m) = (-1)^{mn} (y_1 \wedge \cdots \wedge y_m) \wedge (x_1 \wedge \cdots \wedge x_n)
\]
(just move each $y_i$ past all the $x$s, pick up sign $(-1)^n$ each time)
It's important that not all elements are decomposable!

\( \alpha \in \Lambda^2(U) \) decomposable \( \Rightarrow \alpha \wedge \alpha = 0 \)

but if \( \alpha = e_1 \wedge e_2 + e_3 \wedge e_4 \), then \( \alpha \wedge \alpha = 2 e_1 \wedge e_2 \wedge e_3 \wedge e_4 \neq 0 \)

**Rk** Basis for \( \Lambda(V) \) is \( \{e_1 \wedge \ldots \wedge e_i, \ldots \} \), \( i_1 < i_2 < \ldots < i_n \). Note \( n \leq \dim V \).

Thus \( \Lambda(V) = \bigoplus_{n=0}^{\dim V} \Lambda^n(V) \). \( \dim \Lambda(V) = 2^{\dim V} \).

**Ex** \( \dim V = 3 \):

<table>
<thead>
<tr>
<th>basis</th>
<th>( e_1 )</th>
<th>( e_2 )</th>
<th>( e_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_1 \wedge e_2 )</td>
<td>( e_1 \wedge e_3 )</td>
<td>( e_2 \wedge e_3 )</td>
<td></td>
</tr>
</tbody>
</table>

These constructions are also functorial:

\[
\begin{array}{ccc}
\text{given } & \Phi : V \rightarrow W & \text{then } \\
\Downarrow & \downarrow & \downarrow \\
\text{get } & T\Phi : T(V) \rightarrow T(W) & \text{and } \\
\Lambda \Phi : & \Lambda(V) \rightarrow \Lambda(W) & \text{(by universal property)}
\end{array}
\]

Also \( T\Phi \), \( \Lambda \Phi \) preserve degrees

(e.g. concretely, \( T\Phi(x_1 \wedge \ldots \wedge x_n) = \Phi(x_1) \wedge \ldots \wedge \Phi(x_n) \))

So can take \( \Lambda^k \Phi : \Lambda^k V \rightarrow \Lambda^k W \).

Now say \( V \) 2-dim, \( \Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \)

Then \( \left( \Lambda^2 \Phi \right)(e_1 \wedge e_2) = (ae_1 + ce_2) \wedge (be_1 + de_2) = (ad - bc)e_1 \wedge e_2 \)

So \( \Lambda^2 \Phi : \Lambda^2 V \rightarrow \Lambda^2 W \) is multiplication by \( \det \Phi \)

Similar for any \( V \) (use permutation formula for \( \det \)).

This motivates:

**Def** \( V \) vector space: \( \det V = \Lambda^{\dim V}(V) \).

**Rk** \( \det V \) is a 1-d vector space, but generally has no distinguished basis, except if \( \dim V = 0 \), \( \Lambda^0 V = \mathbb{R} \).

**Def** An orientation of \( V \) is a connected component of \( (\det V) \setminus \{0\} \).

If \( \dim V = 1 \) this means "sense of direction" \( \vec{v} \) vs \( \vec{w} \) (here \( \det V = 1 \))

If \( \dim V = 2 \) it means "sense of rotation" \( e_1 \wedge e_2 \) for \( \vec{e}_1 \wedge \vec{e}_2 \) vs \( \vec{e}_1 \wedge \vec{e}_2 \) or \( \vec{e}_2 \wedge \vec{e}_1 \) (leaves defined the same orientation if can go from one to the other without \( e_1, e_2 \) passing through zero, i.e. without \( e_1, e_2 \) becoming parallel).
Duality

Prop. The pairing \((V^*)^k \times V^k \rightarrow \mathbb{R}\)

given by \((\alpha_1 \otimes \cdots \otimes \alpha_k, v_1 \otimes \cdots \otimes v_k) \rightarrow \alpha_1(v_1) \alpha_2(v_2) \cdots \alpha_k(v_k)\)

and the pairing \(\Lambda^k V^* \times \Lambda^k V \rightarrow \mathbb{R}\)

\((\alpha_1 \wedge \cdots \wedge \alpha_k, v_1 \wedge \cdots \wedge v_k) \rightarrow \det(\alpha_i(v_j))_{i,j=1,\ldots,k}\)

are non-degenerate.

Pf. (For \(V\) finite-dimensional) Take a basis \(e_1, \ldots, e_n\) of \(V\) with dual basis \(e^1, \ldots, e^n\) of \(V^*\).

Then \(\{e^1 \otimes \cdots \otimes e^k\}, \{e_1 \wedge \cdots \wedge e_k\}\) are dual bases of \(V^k\) and \((V^*)^k\).

Similarly \(\{e^1 \wedge \cdots \wedge e^k\}, \{e_1 \wedge \cdots \wedge e_k\}\)

\(\Lambda^k V^* \wedge \Lambda^k V\).

Using this, can view \(\omega \in \Lambda^k V^*\) as an alternating multilinear function:

\[\omega(v_1, \ldots, v_k) = \langle \omega, v_1 \wedge \cdots \wedge v_k \rangle\]

E.g.

\[(\alpha_1 \wedge \alpha_2) (v_1, v_2) = \langle \alpha_1 \wedge \alpha_2, v_1 \wedge v_2 \rangle = \alpha_1(v_1) \alpha_2(v_2) - \alpha_1(v_2) \alpha_2(v_1)\]

Transfer to manifold:

Now suppose \(E\) smooth vector bundle over \(M\).

Then can define vector bundle \(\bigoplus_{n=0}^{\text{rank } E} \Lambda^n E\) with fibers \((\Lambda^n E)_p = \Lambda^n (E_p)\)

[Exercise on next HW to make charts!]

Apply this to \(E = (TM)^*\) (also called \(T^*M\))

\[\Lambda^n TM = \bigoplus_{n=0}^{\text{dim } M} \Lambda^n TM = \bigoplus_{n=0}^{\text{dim } M} \Lambda^n TM\]

\((E = \text{dim } M)\)

Def. \(\Omega(M) = \bigoplus_{n=0}^{\text{dim } M} \Omega^n (M)\) ("differential forms on \(M\)"

\(\Omega(M)\) is a \(\mathbb{Z}\)-graded, graded-commutative algebra.
Ex 1. If $M = \mathbb{A}^3$, $\omega = 3 + dx^1 + dx^2 + dx^3 = 4 dx^1 dx^3 \in \Omega^2(M)$.
2. recall for $f: M \to \mathbb{R}$, $df \in \Gamma^*(T^*M) = \Omega^1(M) \subseteq \Omega^2(M)$, $d(fg) = f dg + g df$

$$f(xy) = x^2 y \quad df = 2xy \, dx + x^2 \, dy$$
3. $\omega$ on $\mathbb{A}^2$, take any $U \subseteq \mathbb{A}^2$ on which pole cords $(r, \theta)$ exist, then $\omega = dx \wedge dy = d(\cos \theta) \wedge d(\sin \theta)$

Can view diff. forms as functionals on vector fields: given $\omega \in \Omega^k(M)$

$$\exists \Phi : \mathfrak{X}(M)^k \to C^\infty(M)^*$$

$$\Phi(\xi_1, \ldots, \xi_k)(\gamma) = \omega(\gamma) \quad \forall \gamma \in C^\infty(M)$$

This has key properties: (1) alternating: $\omega(-\xi_1, \ldots, -\xi_k, \ldots) = -\omega(... \xi_i, \ldots, \xi_j, ...)$

(2) multilinear over $C^\infty(M)$: $\omega(f \xi_1, \ldots, f \xi_k, ...) = f \omega(\xi_1, \ldots, \xi_k, ...)$ \hspace{1cm} $\forall f \in C^\infty(M)$

Indeed any map with these properties is $\Phi$ for some $\omega \in \Omega^k(M)$:

**Lemma (linearity principle)**

E smooth vector bundle over $M$: suppose $\hat{\gamma}: \Gamma(E) \to C^\infty(M)$ has

$$\hat{\gamma}(f \gamma) = f \cdot \hat{\gamma}(\gamma) \quad \forall f \in C^\infty(M), \gamma \in \Gamma(E)$$

Then $\exists \gamma \in \Gamma(E)$ s.t. $\hat{\gamma}(f \gamma)(p) = \gamma(p) \cdot \omega(p)$

**Pf** Exercise on this week's HW.

**C** Any $\Phi: \mathfrak{X}(M)^k \to \mathbb{R}$ which is alternating and $C^\infty(M)$-multilinear is $\Phi$ for some $\omega \in \Omega^k(M)$.

**Pf** Exercise on this week's HW.

With this in mind define $\Phi$ over $\omega$.

**Def/Prop** $f: M \to N$ smooth map, $\omega \in \Omega^k(N)$: $f^\ast \omega \in \Omega^k(M)$ is determined by

$$\left[(f^\ast \omega)(\xi_1, ..., \xi_k)\right](p) = \omega(df_p(\xi_1), ..., df_p(\xi_k)) \quad \xi_1, ..., \xi_k \in \mathfrak{X}(M)$$

It holds $(fg)^\ast \omega = g^\ast f^\ast \omega$, $\quad \left[M \to N \to P \atop f \circ g\right]$

$f^\ast (\alpha \wedge \beta) = f^\ast \alpha \wedge f^\ast \beta$.

**Pf Sketch** To see well def: just check alternating, $C^\infty(M)$-linear. Use chain rule for composition.

For compatibility with wedge product: note our def. is equivalent to saying that

$$[f^\ast \omega](p) = \Lambda(df)^\ast \omega(p), \quad \text{but this is } df = (df_p) \omega(p)$$

Thus if $\omega = \omega_1 \wedge \alpha \omega_2$, $\alpha \in \Omega^1(M)$, we get $f\omega = f^\ast \omega_1 \wedge f^\ast \omega_2$;

the desired $f^\ast (\alpha \omega_2) = f^\ast \alpha \wedge f^\ast \omega_2$ easily follows.
Then there exists a unique sequence of maps
\[ 0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \cdots \]
with the properties
1) \( d \) on \( \Omega^0(M) = C^0(M) \) agrees with the one we already know,
2) \( d \) is linear over \( \mathbb{R} \),
3) \( d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\text{deg} \alpha} \alpha \wedge d\beta \) for \( \alpha \in \Omega^\text{odd}(M) \),
4) \( d^2 = 0 \).

They also obey \( f^* (dw) = d(f^* \omega) \) for any \( f : M \to N \), \( \omega \in \Omega^*(N) \).

Pf
First consider \( \mathbb{A}^n \).
\[ \omega \in \Omega^1(\mathbb{A}^n) \text{ has } \omega = \sum_{i_1, \ldots, i_n} f_{i_1, \ldots, i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n} = \sum_{i_1} f_{x_i} dx^x \]
The properties above then determine \( d\omega = \sum_{i_1, i_2} \frac{\partial f_{i_1, i_2}}{\partial x_{i_1}} dx^{i_2} \wedge dx^{i_1} \wedge dx^x = \sum_{i_1} \frac{\partial f_{x_i}}{\partial x_{i_1}} dx^{i_1} \wedge dx^x \)
and can check directly it obeys 1-4): 1,2) are easy,
3) \[ \alpha = f \ dx^x / \beta = g \ dx^x \quad d(\alpha \wedge \beta) = d(fg) \ dx^x \wedge dx^x = \sum_{i_1} \frac{\partial (fg)}{\partial x_{i_1}} dx^{i_1} \wedge dx^x \wedge dx^x \quad = \sum_{i_1} \frac{\partial f}{\partial x_{i_1}} dx^{i_1} \wedge g \ dx^x + (-1)^{\text{deg} \alpha} f \ dx^x \wedge \frac{\partial g}{\partial x_{i_1}} dx^{i_1} \wedge dx^x = d\alpha \wedge \beta + (-1)^{\text{deg} \alpha} \alpha \wedge d\beta \]
4) \( f_\sharp \Omega^0 = \Omega^0 \), \( d(d\phi) = d\left( \frac{\partial \phi}{\partial x_i} \ dx_i \right) = \frac{\partial^2 \phi}{\partial x_i \ partial x_j} \ dx_i \wedge dx_j = 0 \) (by antiderivatives under \( i \to j \))

Generally, \( d(d(f \ dx^x)) = d(d(f) \ dx^x) = d(f \ dx^x) - d(f \wedge d(dx^x)) = 0 \)

Also check: if \( \psi : U \to V \) and \( \omega \in \Omega^*(V) \) then \( \psi^* (dw) = d(\psi^* \omega) \).
\[ \begin{bmatrix} \psi^* (dx_i) = \frac{\partial \psi_i}{\partial x_j} dx_j + \frac{\partial \psi_{ij}}{\partial x_k} dx_k \wedge dx_j + \frac{\partial \psi_{ijk}}{\partial x_l} dx_l \wedge dx_k \wedge dx_j \\ \psi^* (dx^x) = \sum_{i_1} \frac{\partial \psi_{x_i}}{\partial x_{i_1}} dx^{i_1} \wedge dx^x \end{bmatrix} \quad \text{x cords in V y cords in U} \]

To define \( d\omega \) on general \( M \), then work locally: for any \( p \in M \) fix a chart \((U, x)\)
then say \( d\omega = x^* d(x^* \omega) \)

To check independence of the charts
\[ x_i^* d(x_i^* \omega_j) = x_i^* (\psi_{ij}^*)^* d(\psi_{ij}^* x_i^* \omega_j) = x_i^* d(x_i^* \omega_j) \]

Finally check it obeys 1-4) and pullback property: this follows from the same for \( \mathbb{A}^n \) alone.
Ex \( M = \mathbb{A}^3 \): 
\[ d(x^1 x^2 \, dx^2) = dx^1 \wedge (x^2 \, dx^3) + x^1 \, dx^2 \wedge dx^8 = x^2 \, dx^1 \wedge dx^2 \]