Exterior algebra

Def 1) An *algebra* is a vector space A with an associative product $A\times A \to A$ and unit $1\in A$.

2) An alg. A is *$\mathbb{Z}$-graded* if $A = \bigoplus_{n=-\infty}^{\infty} A_n$ and $A_n\cdot A_m \subset A_{n+m}$

Elements $x\in A_n$ are called *homogeneous of degree n*. Write this $|x|=n$.

3) A $\mathbb{Z}$-graded alg A is *(graded) commutative* if $x\cdot y = (-1)^{|x||y|}y\cdot x$ for x,y homogeneous.

Rk $1\in A_0$.

Def V vector space: the *tensor algebra* $T(V)$ is $\mathbb{Z}$-graded alg $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$

$$= \mathbb{R}\oplus V\oplus V\otimes V\oplus\cdots$$

with product $V^{\otimes m}\times V^{\otimes n}\longrightarrow V^{\otimes m+n}$ given by the standard bilinear map

(recall for any vector spaces V, W $\exists$ $V\times W \to V\otimes W$ bilinear)

Prop (Universal property of $T(V)$) If A is any algebra and $j: V\to A$ any linear map, then $\exists!$ algebra hom $\tilde{j}: T(V)\to A$ s.t. the diagram

$$\begin{tikzcd} T(V) \arrow[dr, "\tilde{j}"] & \\ & A \\ V \arrow[uu, hook, "\iota"] \arrow[ur, "j"'] & \end{tikzcd}$$

commutes, where $\iota: V\hookrightarrow T(V)$ is the obvious inclusion.

Pf Uniqueness:

Commutativity forces $\tilde{j}(x) = j(x)$ for $x\in V$. These elements generate $T(V)$. Thus algebra hom property determines $\tilde{j}$ on the rest of $T(V)$.

Existence:

$$j_n: \begin{array}{ccc} V^n & \longrightarrow & A \\ (x_1,\dots,x_n) & \mapsto & j(x_1)\cdots j(x_n)\end{array}$$

is multilinear, so factors thru

$$\begin{tikzcd} & V^{\otimes n} \arrow[dr, "\tilde{j}_n"] & \\ V^n \arrow[ur] \arrow[rr, "j_n"'] & & A \end{tikzcd}$$

then define $\tilde{j}$ to be the linear map agreeing with $\tilde{j}_n$ on $V^{\otimes n}\subset T(V)$,

check $\tilde{j}(x_1\otimes\cdots\otimes x_n)\tilde{j}(y_1\otimes\cdots\otimes y_m) = j(x_1)\cdots j(x_n)\,j(y_1)\cdots j(y_m) = \tilde{j}(x_1\otimes\cdots\otimes x_n\otimes y_1\otimes\cdots\otimes y_m)$

Rk This universal property determines $T(V)$ up to unique $\simeq$:
if have another $T'(V)$ with the same property, then get the diagram

$$\begin{tikzcd} & T(V) \arrow[d, "\varphi"] \\ V \arrow[ur, "\iota"] \arrow[r, "\iota'"] \arrow[dr, "\iota"'] & T'(V) \arrow[d, "\varphi'"] \\ & T(V) \end{tikzcd}$$

and the uniqueness part of the univ. prop. shows $\varphi' \circ \varphi = 1$, so φ is $\simeq$.

Rk If $\{e_i\}$ basis for V, $\dim V = k$
then $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}$ basis for $V^{\otimes n}$, $\dim V^{\otimes n} = k^n$

Def V vector space: exterior algebra $\Lambda(V)$ is $T(V)/I$ where I is 2-sided ideal generated by $\{v \otimes v : v \in V\}$.

Prop (Universal property of $\Lambda(V)$) If A is any algebra and $j: V \to A$ any linear map s.t. $j(x)^2 = 0$ $\forall x \in V$, $\exists!$ alg hom $\tilde{j}: \Lambda(V) \to A$ s.t.

$$\begin{tikzcd} \Lambda(V) \arrow[dr, "\tilde{j}"] & \\ & A \\ V \arrow[uu, hook] \arrow[ur, "j"] & \end{tikzcd}$$

commutes.

Pf We have $\tilde{j}: T(V) \to A$ by universal property, and it descends to $\Lambda(V)$ since $\tilde{j}(x^2) = \tilde{j}(x)^2 = j(x)^2 = 0$. Uniqueness also follows from the uniqueness for $T(V) \to A$. ∎

Notation Write the product in $\Lambda(V)$ as $\wedge$.

Prop $\Lambda(V)$ is $\mathbb{Z}$-graded and (graded) commutative.

Pf $\mathbb{Z}$-grading follows from fact that I is generated by homogeneous elements. For commutativity:
First note for $x, y \in V$ we have $(x+y) \wedge (x+y) = 0$, so $\underbrace{\cancel{(x \wedge x)}}_{0} + \underbrace{\cancel{(y \wedge y)}}_{0} + (x \wedge y) + (y \wedge x) = 0$, so $x \wedge y = -y \wedge x$.
Enough to check on decomposable elements:

$$(x_1 \wedge \cdots \wedge x_n) \wedge (y_1 \wedge \cdots \wedge y_m) = (-1)^{mn} (y_1 \wedge \cdots \wedge y_m) \wedge (x_1 \wedge \cdots \wedge x_n)$$

(just move each y past all the x's, pick up sign $(-1)^n$ each time) ∎

Rk It's important that not all elements are decomposable!

e.g. $\alpha \in \Lambda^2(V)$ decomposable $\Rightarrow$ $\alpha \wedge \alpha = 0$

but if $\alpha = e_1 \wedge e_2 + e_3 \wedge e_4$ then $\alpha \wedge \alpha = 2\, e_1 \wedge e_2 \wedge e_3 \wedge e_4 \neq \underline{\underline{0}}$

Rk Basis for $\Lambda(V)$ is $e_{i_1} \wedge \cdots \wedge e_{i_n}$ $\quad i_1 < i_2 < \cdots < i_n$. Note $n \leq \dim V$.

Thus $\Lambda(V) = \bigoplus_{n=0}^{\dim V} \Lambda^n(V)$. $\quad \dim \Lambda(V) = 2^{\dim V}$.

Ex $\dim V = 3$: basis

$$\begin{matrix} & e_1 & e_1 \wedge e_2 & \\ 1 & e_2 & e_1 \wedge e_3 & e_3 \\ & e_3 & e_2 \wedge e_3 & \end{matrix}$$

These constructions are also <u>functorial</u>:

given $\varphi : V \longrightarrow W$

get $T\varphi : T(V) \to T(W)$

and $\Lambda\varphi : \Lambda(V) \longrightarrow \Lambda(W)$

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d] & W \arrow[d] \\ T(V) \arrow[r, "T\varphi"] & T(W) \end{tikzcd}$$

(by universal property)

Also $T\varphi$, $\Lambda\varphi$ preserve degrees

(e.g. concretely, $T\varphi(x_1 \otimes \cdots \otimes x_n) = \varphi(x_1) \otimes \cdots \otimes \varphi(x_n)$) $\qquad$ So can take $\Lambda^k\varphi : \Lambda^k V \to \Lambda^k W$.

Now say V 2-dim, with basis $\{e_1, e_2\}$ $\quad \varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad$ Then

$$\begin{aligned} (\Lambda^2\varphi)(e_1 \wedge e_2) &= (ae_1 + ce_2) \wedge (be_1 + de_2) \\ &= (ad - bc)\, e_1 \wedge e_2 \\ &= (\det \varphi)\, e_1 \wedge e_2 \end{aligned}$$

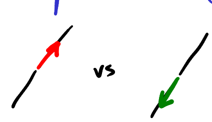
ie, $\Lambda^2\varphi : \Lambda^2 V \to \Lambda^2 V$ is <u>multiplication</u> by $\det \varphi$

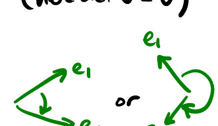
Similar for any V (use permutation formula for det). This motivates:

Def V vector space: $\det V = \Lambda^{\dim V}(V)$.

Rk $\det V$ is a 1-d vector space, but generally has no distinguished basis, <u>except</u> if $\dim V = 0$!
$(\Lambda^0 V = \mathbb{R})$

Def An <u>orientation</u> of V is a connected component of $(\det V) \setminus \{0\}$.

If $\dim V = 1$ this means "sense of direction"  (here $\det V = V$)

If $\dim V = 2$ it means "sense of rotation": $e_1 \wedge e_2$ for  vs 

(bases define the same orientation if can go from one to the other without $e_1 \wedge e_2$ passing thru zero, i.e. without e_1, e_2 becoming parallel)

Duality

Prop The pairing $(V^*)^{\otimes k} \times V^{\otimes k} \to \mathbb{R}$

given by $(\alpha_1 \otimes \cdots \otimes \alpha_k, v_1 \otimes \cdots \otimes v_k) \mapsto \alpha_1(v_1)\alpha_2(v_2)\cdots\alpha_k(v_k)$

and the pairing $\Lambda^k V^* \times \Lambda^k V \to \mathbb{R}$

$$(\alpha_1 \wedge \cdots \wedge \alpha_k, v_1 \wedge \cdots \wedge v_k) \mapsto \det\left(\alpha_i(v_j)\right)_{i,j=1,\dots,k}$$

are nondegenerate.

Pf (For V finite-dimensional) Take a basis $e_1, \dots, e_n$ of V with dual basis $e^1, \dots, e^n$ of V^*.

Then $\{e^{i_1} \otimes \cdots \otimes e^{i_k}\}$, $\{e_{i_1} \otimes \cdots \otimes e_{i_k}\}$ are dual bases of $V^{\otimes k}$ and $(V^*)^{\otimes k}$.

Similarly $\{e^{i_1} \wedge \cdots \wedge e^{i_k}\}$, $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ " " " " $\Lambda^k V$ " $\Lambda^k V^*$. ∎

Using this, can view $\omega \in \Lambda^k V^*$ as an alternating multilinear functional:

$$\omega(v_1, \dots, v_k) = \langle \omega, v_1 \wedge \cdots \wedge v_k \rangle$$

e.g. $(\alpha_1 \wedge \alpha_2)(v_1, v_2) = \langle \alpha_1 \wedge \alpha_2, v_1 \wedge v_2 \rangle = \alpha_1(v_1)\alpha_2(v_2) - \alpha_1(v_2)\alpha_2(v_1)$

Transfer to manifold:

Now suppose E smooth vector bundle over M.

Then can define vector bundle $\Lambda E = \bigoplus_{n=0}^{\operatorname{rank} E} \Lambda^n E$ with fibers $(\Lambda^\cdot E)_p = \Lambda^\cdot(E_p)$

[Exercise on next HW to make charts!]

Apply this to $E = (TM)^*$ (also called T^*M)

$$\Lambda T^*M = \underbrace{\Lambda^0 T^*M}_{\cong\, M \times \mathbb{R}} \oplus \underbrace{\Lambda^1 T^*M}_{\cong\, T^*M} \oplus \cdots \oplus \underbrace{\Lambda^m T^*M}_{=\, \det T^*M}$$

$(m = \dim M)$

$\det T^*M$ (bundle of rank 1, i.e. line bundle)

Def $\Omega(M) = \Gamma(\Lambda T^*M) = \bigoplus_{n=0}^{\dim M} \Omega^n(M)$ ("differential forms on M")

$\Omega(M)$ is a $\mathbb{Z}$-graded, graded-commutative algebra.

Ex ① if $M=\mathbb{A}^3$, $\omega = 3 + dx^1 + dx^2\wedge dx^3 - 4\,dx^1\wedge dx^3 \in \Omega(M)$.

② recall for $f: M\to\mathbb{R}$, $df \in \Gamma(T^*M) = \Omega^1(M) \subset \Omega(M)$, $d(fg) = f\,dg + g\,df$

e.g. $f: \mathbb{A}^2 \to \mathbb{R}$

$f(x,y) = x^2y \qquad df = 2xy\,dx + x^2\,dy$

③ if $M = \mathbb{A}^2$, take any $U\subset\mathbb{A}^2$ on which polar coords (r,θ) exist, then $\omega = dx\wedge dy = d(r\cos\theta)\wedge d(r\sin\theta)$
$= (\cos\theta\,dr - r\sin\theta\,d\theta)\wedge(\sin\theta\,dr + r\cos\theta\,d\theta)$
$= r\,dr\wedge d\theta$

Can view diff. forms as functionals on vector fields: given $\omega\in\Omega^k(M)$

have a map $\tilde\omega: \mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M) \longrightarrow C^\infty(M)$

$$(\xi_1, \ldots, \xi_k) \longmapsto \left[p \mapsto \omega(p)\,(\xi_1(p), \xi_2(p), \ldots, \xi_k(p))\right]$$

This has key properties ① alternating: $\tilde\omega(\ldots,\xi_i,\ldots,\xi_j,\ldots) = -\tilde\omega(\ldots,\xi_j,\ldots,\xi_i,\ldots)$

② multilinear over $C^\infty(M)$: $\tilde\omega(f\xi_1,\xi_2,\ldots) = f\cdot\tilde\omega(\xi_1,\xi_2,\ldots)$ $\quad\forall f\in C^\infty(M)$

Indeed any map with these properties is $\tilde\omega$ for some $\omega\in\Omega^k(M)$:

Lemma (Linearity principle)

E smooth vector bundle over M: suppose $\tilde\eta: \Gamma(E)\longrightarrow C^\infty(M)$ has

$$\tilde\eta(f\cdot s) = f\cdot\tilde\eta(s) \qquad \forall f\in C^\infty(M),\ s\in\Gamma(E)$$

Then $\exists\,\eta\in\Gamma(E^*)$ s.t. $\left[\tilde\eta(f\cdot s)\right](p) = \eta(p)\cdot s(p)$

Pf Exercise on this week's HW.

Cor Any $\phi: \mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)\longrightarrow\mathbb{R}$ which is alternating and $C^\infty(M)$-multilinear is $\tilde\omega$ for some $\omega\in\Omega^k(M)$.

Pf Exercise on this week's HW.

With this in mind often drop the $\sim$ over ω.

Def/Prop $f: M\longrightarrow N$ smooth map, $\omega\in\Omega^k(N)$: $f^*\omega\in\Omega^k(M)$ is determined by

$$\left[\left(\widetilde{f^*\omega}\right)(\xi_1,\ldots,\xi_k)\right](p) = \omega\left(df_p(\xi_1(p)), \ldots, df_p(\xi_k(p))\right) \qquad \xi_1,\ldots,\xi_k\in\mathfrak{X}(M)$$

It obeys $(f\circ g)^*\omega = g^*f^*\omega$,

$$\begin{array}{ccccc} g^*f^*\omega & & f^*\omega & & \omega \end{array}$$
$$\begin{tikzcd} M \arrow[r, "g"] \arrow[rr, bend right, "f\circ g"'] & N \arrow[r, "f"] & P \end{tikzcd}$$

$f^*(\alpha\wedge\beta) = f^*\alpha\wedge f^*\beta$.

Pf Sketch To see well def: just check alternating, $C^\infty(M)$-linear. Use chain rule for composition.

For compatibility with wedge product: note our def. is equivalent to saying that

$[f^*\omega](p) = [\Lambda(df_p)]^*(\omega(p))$, but this is $= \Lambda(df_p^*)(\omega(p))$ $\quad$ [in general if $A: V\to W$ then $\Lambda A: \Lambda V\to\Lambda W$; $A^*: W^*\to V^*$, $\Lambda(A^*): \Lambda W^*\to\Lambda V^*$, $\Lambda(A^*) = (\Lambda A)^*$]

This if $\omega = \alpha_1\wedge\cdots\wedge\alpha_k$, $\alpha_i\in\Omega^1(N)$, we get $f^*\omega = f^*\alpha_1\wedge\cdots\wedge f^*\alpha_k$;

the desired $f^*(\alpha\wedge\beta) = f^*\alpha\wedge f^*\beta$ easily follows. ∎

Ex $f: \mathbb{A}^m \to \mathbb{A}^n$, $(y^1,\dots,y^m) \mapsto (x^1,\dots,x^n)$ $\qquad f^*(dx^i) = \sum_{j=1}^{m} \frac{\partial x^i}{\partial y^j}\, dy^j$

$\left[\text{pf: } [f^*(dx^i)](\partial/\partial y^j) = dx^i(df(\partial/\partial y^j)) = dx^i\left(\frac{\partial x^k}{\partial y^j}\frac{\partial}{\partial x^k}\right) = \frac{\partial x^i}{\partial y^j}\right]$

[more precisely this should be written $\frac{\partial(x^i\circ f)}{\partial y^j}$]

Thm There exists a unique sequence of maps

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots$$

with the properties

1) d on $\Omega^0(M) \simeq C^\infty(M)$ agrees with the one we already know,
2) d is linear over $\mathbb{R}$,
3) $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^{|\alpha|}\alpha\wedge d\beta$ $\quad$ for $\alpha\in\Omega^{|\alpha|}(M)$,
4) $d^2 = 0$.

They also obey $f^*(d\omega) = d(f^*\omega)$ for any $f: M\to N$, $\omega\in\Omega(N)$.

Pf First consider $\mathbb{A}^m$. $\qquad [I = (i_1,\dots,i_k),\ dx^I = dx^{i_1}\wedge\dots\wedge dx^{i_k}]$

$\omega\in\Omega^k(\mathbb{A}^m)$ has $\omega = \sum_{i_1\dots i_k} f_{i_1\cdots i_k}\, dx^{i_1}\wedge\dots\wedge dx^{i_k} = \sum_I f_I\, dx^I$

The properties above then <u>determine</u> $d\omega = \sum_{i_1,\dots,i_k,j} \frac{\partial f_{i_1\cdots i_k}}{\partial x^j}\, dx^j\wedge dx^{i_1}\wedge\dots\wedge dx^{i_k} = \sum_{I,j}\frac{\partial f_I}{\partial x^j}\, dx^j\wedge dx^I$

and can check directly it obeys 1)-4): 1), 2) are easy,

3) $\quad \alpha = f\,dx^I \quad \beta = g\,dx^J \quad d(\alpha\wedge\beta) = d(fg\, dx^I\wedge dx^J) = \frac{\partial(fg)}{\partial x^k}\, dx^k\wedge dx^I\wedge dx^J$

$$= \frac{\partial f}{\partial x^k}dx^k\wedge dx^I\wedge g\,dx^J + (-1)^{|I|} f\,dx^I\wedge\frac{\partial g}{\partial x^k}dx^k\wedge dx^J = d\alpha\wedge\beta + (-1)^{|\alpha|}\alpha\wedge d\beta$$

4) for $f\in\Omega^0$, $d(df) = d\left(\frac{\partial f}{\partial x^i}dx^i\right) = \frac{\partial^2 f}{\partial x^i\partial x^j}\, dx^i\wedge dx^j = 0$ $\quad$ (by antisymmetry under $i\leftrightarrow j$)

generally, $d(d(f\,dx^I)) = d(df\wedge dx^I) = \cancel{d(df)}\wedge dx^I - df\wedge\cancel{d(dx^I)} = 0$

Also check: if $\varphi: U \to V$ and $\omega\in\Omega(V)$ then $\varphi^*(d\omega) = d(\varphi^*\omega)$.
$\qquad\qquad U\subset\mathbb{A}^m$, $V\subset\mathbb{A}^n$

$\Big[$ $\varphi^*(d(\alpha_i\, dx^i)) = \varphi^*\left(\frac{\partial\alpha_i}{\partial x^j}\, dx^j\wedge dx^i\right) = \frac{\partial\alpha_i}{\partial x^j}\frac{\partial x^j}{\partial y^k}\frac{\partial x^i}{\partial y^l}\, dy^k\wedge dy^l = \frac{\partial\alpha_i}{\partial y^k}\frac{\partial x^i}{\partial y^l}\, dy^k\wedge dy^l$

$d(\varphi^*(\alpha_i\, dx^i)) = d\left(\alpha_i\frac{\partial x^i}{\partial y^j}dy^j\right) = \frac{\partial\alpha_i}{\partial y^k}\frac{\partial x^i}{\partial y^j}\, dy^k\wedge dy^j + \alpha_i\cancel{\frac{\partial^2 x^i}{\partial y^j\partial y^l}\, dy^l\wedge dy^j}$ $\qquad$ x coords in V, y coords in U

and then for ω of general degree, use <u>induction</u> and $\varphi^*(\alpha\wedge\beta) = \varphi^*\alpha\wedge\varphi^*\beta$. $\Big]$

To define $d\omega$ on general M then, work locally: for any $p\in M$ fix a chart (U,x)

then say
$$d\omega = x^* d(x^{-1*}\omega)$$

To check independence of the charts:

$$x_i^* d(x_i^{-1*}\omega_{ij}) = x_i^*(\varphi_{ij}^{-1})^* d(\varphi_{ij}^* x_i^{-1*}\omega_{ij}) \qquad \varphi_{ij} = x_i\circ x_j^{-1}: \mathbb{A}^m\to\mathbb{A}^m$$
$$= x_j^* d(x_j^{-1*}\omega_{ij}) \quad \checkmark$$

Finally check it obeys 1)-4) and pullback property: this follows from the same for $\mathbb{A}^m$ above ∎

Ex $M = \mathbb{A}^3$: $d(x^1 x^2\, dx^2) = dx^1 \wedge (x^2\, dx^2) + x^1\, dx^2 \wedge dx^2 = x^2\, dx^1 \wedge dx^2$