Stokes' Theorem

Motivating example: \( I = [a,b] \)
\[
\int_I \, df = \int_a^b f'(x) \, dx = f(b) - f(a) = \int_{\partial I} f
\]

Theorem (Stokes' Theorem): \( M \) a manifold with boundary, \( \omega \in \Omega^{m-1} (M) \):
\[
\int_M \, d\omega = \int_{\partial M} \omega
\]

Proof (Pf): Take a partition of unity \( \rho_i \) relative to covering by charts \( (U_i, x_i) \). Let \( \omega_i = \rho_i \omega \).
Thus reduce to case where \( \supp \omega_i \subset U_i \) for \( (U_i, x_i) \) a chart.

Then, \( \int_M \, d\omega = \sum_i \int_{x(U_i)} \omega_i \, \gamma = \sum_i \int_{x(U_i)} \omega_i \, d\gamma \) where \( \gamma = (x^{-1})^* \omega = f_i \, dx^2 \)
\[
d\gamma = \frac{\partial f_i}{\partial x^j} \, dx^j \wedge dx^1
\]

If \( U \) is an interior chart:
\[
\int_M \, d\omega = \sum_i \int_{x(U_i)} \omega_i \, d\gamma
\]
\[
= \sum_i \int_{x(U_i)} \frac{\partial f_i}{\partial x^j} \, dx^j \wedge dx^1 \wedge \ldots \wedge dx^m = 0
\]

If \( U \) is a boundary chart:
\[
\int_M \, d\omega = \sum_i \int_{x(U_i)} \omega_i \, \gamma = \int_{\partial M} \omega
\]
\[
= \sum_i \int_{\partial M} \omega_i
\]

(With the induced orientation on \( \partial M \) — since \( dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m \) is \( \varepsilon \)-oriented on \( M \) so that \( dx^1 \wedge \ldots \wedge dx^m \) is \( \varepsilon \)-oriented on \( \partial M \) )

Example (Ex):
\[
\omega = f \, dx + g \, dy
\]
\[
d\omega = \left( \frac{\partial^2 g}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \, dx \wedge dy
\]
\[
\int_M \, d\omega = \int_{\partial M} \omega = \int \omega = \int f \, dx + g \, dy
\]

This is standard "Stokes' Theorem": for a vector field \( \mathbf{v} = [f, g] \) in \( \mathbb{R}^2 \), \( \text{curl} \mathbf{v} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \),
\[
\int_M \, \text{curl} \mathbf{v} \, dA = \int_{\partial M} \mathbf{v} \cdot d\mathbf{s}
\]
\[ \omega = \nabla \cdot \mathbf{v} = \int \mathrm{div} \, \mathbf{v} = \int_{\partial M} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}S \]

This is "divergence theorem": \( \mathbf{v} = [f, g, h] \) in \( \mathbb{R}^3 \),
\[ \mathrm{div} \, \mathbf{v} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \quad \mathbf{n} \, \mathrm{d}S = [dxdy, dzdx, dxzd] \]

\[ \int_{\partial M} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}S = \int_{\partial M} \mathrm{div} \, \mathbf{v} \, \mathrm{d}S \]

**Case:** If \( \partial M = \emptyset \) and \( \omega \mathcal{H}^m(M) \) then \( \int_{M} \omega = 0 \).

Let's practice integrating a bit.

**Ex:** \( M = S' \).

View \( S' \) as \( [0, \pi] \times \{0 \leq \theta \leq 2\pi\} \). Fix some \( f: \mathbb{R} \rightarrow \mathbb{R} \), with \( f(2\pi \theta) = f(\theta) \).

\( f(t) \, dt \) on \( (0, 2\pi) \) extends to an element of \( \Omega'(S') \). To see this explicitly, cover by 2 charts \( U_1, U_2 \)

\[ U_1 = \{(0, 2\pi)\} \]
\[ U_2 = \{[0, \pi] \times (2\pi-\pi, 2\pi]\} \]

Coordinates:
\[ x_1: U_1 \rightarrow \mathbb{R} \]
\[ x_1(t) = t \]
\[ x_2: U_2 \rightarrow \mathbb{R} \]
\[ x_2(t) = \begin{cases} t & \text{if } t \leq \pi \\ t + 2\pi & \text{if } t > \pi \end{cases} \]

Then define \( \omega = \left\{ \begin{array}{ll} f(x_1) \, dx_1 & \text{on } U_1 \\ f(x_2) \, dx_2 & \text{on } U_2 \end{array} \right. \). They agree on \( U_1 \cap U_2 \) since here \( dx_1 = dx_2 = dt \) and \( f(x_1) = f(x_2) \) using periodicity.

Now to integrate:
\[ x_1^{-1}(f(x_1) \, dx_1) = f(x) \, dx \quad x_2^{-1}(f(x_2) \, dx_2) = f(x) \, dx \]
\[ \Omega'_1((0, 2\pi)) \quad \sup \, x \in U_1, \quad \Omega'_2((2\pi-\pi, 2\pi+\pi)) \]
\[ x_1^{-1} \rho_1(x) = \left\{ \begin{array}{ll} \rho_1(x) & \text{for } x \in (0, 2\pi) \\ \rho_1(x-2\pi) & \text{for } x \in [2\pi, 2\pi+\pi] \end{array} \right. \]
\[ x_2^{-1} \rho_2(x) = \left\{ \begin{array}{ll} \rho_2(x) & \text{for } x \in (2\pi-\pi, 2\pi] \\ \rho_2(x-2\pi) & \text{for } x \in [2\pi, 2\pi+\pi] \end{array} \right. \]

Then \( \int_{M} \omega = \int_{0}^{2\pi} \rho_1(x) \, f(x) \, dx + \int_{2\pi-\pi}^{2\pi} \rho_2(x) \, f(x) \, dx + \int_{2\pi}^{2\pi+\pi} \rho_2(x-2\pi) \, f(x) \, dx \)
\[ = \int_0^{2\pi} [\rho_1(x) + \rho_2(x)] f(x) \, dx \]
\[ = \int_0^{2\pi} f(x) \, dx \]

Just as we would naively have expected!