Stokes Theorem

Motivating example: $I = [a,b]$

$f: I \to \mathbb{R}$

$df = f'(x)\,dx \qquad \displaystyle\int_I df = \int_a^b f'(x)\,dx = f(b) - f(a) = \int_{\partial I} f$

Thm M mfd with boundary, $\omega \in \Omega_c^{m-1}(M)$: $\displaystyle\int_M d\omega = \int_{\partial M} \iota^* \omega \qquad \iota: \partial M \to M$

Pf Take a partition of unity ρ_i relative to covering by charts (U_i, x_i). Let $\omega_i = \rho_i \omega$.

Thus reduce to case where $\operatorname{supp} \omega \subset U$ for (U,x) a chart.

Then, $\displaystyle\int_M d\omega = \varepsilon \int_U d\omega = \varepsilon \int_{x(U)} (x^{-1})^* d\omega = \varepsilon \int_{x(U)} d\eta$ where $\eta = (x^{-1})^* \omega = f_I\, dx^I$

$$d\eta = \frac{\partial f_I}{\partial x^j}\, dx^j \wedge dx^I$$

If U is an interior chart: $\displaystyle\int_M d\omega = \varepsilon \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \pm \frac{\partial f_I}{\partial x^j}\, dx^1 \cdots dx^m$

$$= \varepsilon \int_{-\infty}^{\infty} \cdots \left[\int_{-\infty}^{\infty} \frac{\partial f_I}{\partial x^j}\, dx^j \right] dx^1 \cdots \widehat{dx^j} \cdots dx^m = \underline{\underline{0}}$$

If U is a boundary chart: $\displaystyle\int_M d\omega = \varepsilon \int_{-\infty}^{\infty} \cdots \left[\int_{-\infty}^{0} \frac{\partial f_I}{\partial x^1}\, dx^1 \right] dx^2 \cdots dx^m$

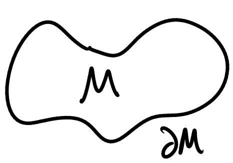
$$= \varepsilon \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_I(x^1 = 0)\; dx^2 \cdots dx^m$$

$$= \int_{\partial M} f_I(x^1 = 0)\; \varepsilon\, dx^2 \wedge \cdots \wedge dx^m$$

(with the induced orient^n on ∂M — since $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m$ is ε-oriented on M so that $dx^2 \wedge \cdots \wedge dx^m$ is ε-oriented on ∂M)

Ex $\omega = f\,dx + g\,dy$

$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy$



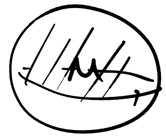
$$\int_M \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy = \int_M d\omega = \int_{\partial M} \omega = \int_{\partial M} f\,dx + g\,dy$$

This is standard "Stokes Thm": for a vector field $\vec{v} = [f \;\; g]$ in $\mathbb{A}^2$, $\operatorname{curl} \vec{v} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$,

and $\displaystyle\int_M \operatorname{curl} \vec{v}\, dA = \int_{\partial M} \vec{v} \cdot d\vec{s}$.

Ex $\omega = f\, dy\wedge dz + g\, dz\wedge dx + h\, dx\wedge dy$

$$d\omega = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx\wedge dy\wedge dz$$



$$\int_M \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx\wedge dy\wedge dz = \int_M d\omega = \int_{\partial M} \omega = \int_{\partial M} f\, dy\wedge dz + g\, dz\wedge dx + h\, dx\wedge dy$$

This is "divergence theorem": $\vec{v} = [f\ g\ h]$ in $\mathbb{A}^3$,

$$\operatorname{div} \vec{v} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \qquad \vec{n}\, dS = [dy\wedge dz,\ dz\wedge dx,\ dx\wedge dy]$$

$$\int_M \operatorname{div} \vec{v} = \int_{\partial M} \vec{v}\cdot\vec{n}\, dS$$

Cor If $\partial M = \phi$ and $\omega \in \Omega^{m-1}(M)$ then $\int_M d\omega = 0$.

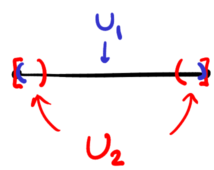
Let's practice integration a bit.

Ex $M = S^1$.

View S^1 as $[0, \overset{t}{\hat{2\pi}}]/\{0\sim 2\pi\}$. Fix some $f: \mathbb{A}^1 \to \mathbb{R}$, with $f(t+2\pi) = f(t)$.

$f(t)\, dt$ on $(0,2\pi)$ extends to an element of $\Omega^1(S^1)$. To see this explicitly, cover by 2 charts U_1, U_2

$$U_1 = \{(0, 2\pi)\}$$
$$U_2 = \{[0,\varepsilon) \cup (2\pi-\varepsilon, 2\pi]\}$$



coords $x_1: U_1 \to \mathbb{A}^1$

$\quad x_1(t) = t$

$x_2: U_2 \to \mathbb{A}^1$

$$x_2(t) = \begin{cases} t & \text{if } t > \pi \\ t+2\pi & \text{if } t<\pi \end{cases}$$

Then define $\omega = \begin{cases} f(x_1)\, dx_1 & \text{on } U_1 \\ f(x_2)\, dx_2 & \text{on } U_2 \end{cases}$. They agree on $U_1 \cap U_2$ since here $dx_1 = dx_2 = dt$ and $f(x_1) = f(x_2)$ using periodicity

Now to integrate:

$$x_1^{-1*}(f(x_1)\, dx_1) = f(x)\, dx \qquad x_2^{-1*}(f(x_2)\, dx_2) = f(x)\, dx$$

partition of unity: $\rho_1 + \rho_2 = 1$

$\operatorname{supp} \rho_1 \subset U_1$

$\operatorname{supp} \rho_2 \subset U_2$

$$\underset{\downarrow}{\Omega^1((0,2\pi))} \qquad \underset{\downarrow}{\Omega^1((2\pi-\varepsilon, 2\pi+\varepsilon))}$$

$$[x_1^{-1*}\rho_1](x) = \rho_1(x) \text{ for } x\in(0,2\pi) \qquad [x_2^{-1*}\rho_2](x) = \begin{cases} \rho_2(x) & \text{for } x\in(2\pi-\varepsilon, 2\pi] \\ \rho_2(x-2\pi) & \text{for } x\in[2\pi, 2\pi+\varepsilon) \end{cases}$$

Then $$\int_M \omega = \int_0^{2\pi} \rho_1(x) f(x)\, dx + \int_{2\pi-\varepsilon}^{2\pi} \rho_2(x) f(x)\, dx + \int_{2\pi}^{2\pi+\varepsilon} \rho_2(x-2\pi) f(x)\, dx$$

$$= \int_0^{2\pi} [\rho_1(x) + \rho_2(x)]\, f(x)\, dx$$

$$= \int_0^{2\pi} f(x)\, dx$$

Just as we would naively have expected!