# de Rham cohomology

$M$ smooth: $\quad 0 \xrightarrow{d_{-1}} \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \to \cdots \xrightarrow{d_{m-1}} \Omega^m(M) \xrightarrow{d_m} 0$

Def ① $\omega \in \Omega^k(M)$ is closed if $d\omega = 0$.

② " " exact if $\exists \alpha \in \Omega^{k-1}(M)$ s.t. $\omega = d\alpha$.

③ $H^k_{dR}(M) = \dfrac{\ker d_k}{\operatorname{im} d_{k-1}} = \dfrac{\Omega^k_{cl}(M)}{\Omega^k_{ex}(M)}$. Vector space.

④ $b_k(M) = \dim H^k_{dR}(M)$ ("Betti numbers").

Rk $H^*_{dR}(M) = \bigoplus_k H^k_{dR}(M)$ is naturally a ring: $([\alpha],[\beta]) \mapsto [\alpha \wedge \beta]$ (Exercise: well defined!)

Ex $H^0_{dR}(M) = \ker d_0 = \{f \in \Omega^0(M) \mid df = 0\}$

Such an $f$ is constant on each connected component of $M$.

So $b_0(M) = \#$ connected components of $M$.

Ex Say $M = S^1$. $H^1_{dR}(M) = \dfrac{\ker d_1}{\operatorname{im} d_0} = \dfrac{\Omega^1(S^1)}{\{d\alpha \mid \alpha \in \Omega^0(S^1)\}}$

Say $\omega \in \Omega^1(S^1)$.

If $\int_{S^1} \omega \neq 0$ then we can't have $\omega = d\alpha$.

But if $\int_{S^1} \omega = 0$ then we do: indeed, write $\omega = f(t)\,dt$, $f: \mathbb{A}^1 \to \mathbb{R}$ periodic
$f(t+2\pi) = f(t)$

$$\int_{S^1} \omega = \int_0^{2\pi} f(t)\,dt = 0$$

then define $\alpha(t) = \int_0^t f(t')\,dt'$. $\alpha(t)$ is periodic, 
$$\alpha(t+2\pi) = \int_0^{2\pi} f(t')\,dt' + \int_{2\pi}^{t+2\pi} f(t')\,dt' = 0 + \int_0^t f(t')\,dt' = \alpha(t),$$
so $\alpha: S^1 \to \mathbb{R}$, $\alpha \in \Omega^0(S^1)$

and $d\alpha = \frac{\partial \alpha}{\partial t}\,dt = f(t)\,dt = \omega$.

Thus the map $\int_{S^1} : \Omega^1(S^1) \to \mathbb{R}$ has kernel $\operatorname{im}(d_0)$, and thus

descends to an isomorphism $\int_{S^1} : H^1_{dR}(S^1) \xrightarrow{\sim} \mathbb{R}$

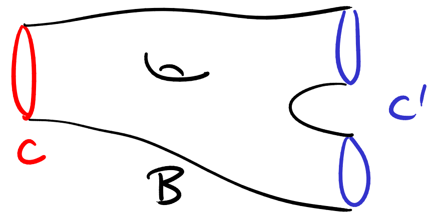
So, $b_1(S^1) = 1$.

More generally: given any oriented manifold $C$, $\dim C = k$, with $\varphi: C \to M$:

- $\int_C : \Omega^k_{closed}(M) \to \mathbb{R}$ factors through $H^k_{dR}(M) \to \mathbb{R}$

$$\omega \mapsto \int_C \varphi^* \omega$$

- Say $(C,\varphi)$ is bordant to $(C',\varphi')$ if have




$\partial B = -C \sqcup C'$

$(C,\varphi) \sim (C',\varphi')$

with $\Phi: B \to M$ $\quad \Phi|_C = \varphi$, $\Phi|_{C'} = \varphi'$

This defines oriented bordism group $\Omega^{SO}_k(M) = \{(C,\varphi) \text{ as above}\}/\sim$

(group operation: $\sqcup$)

If $(C,\varphi) \sim (C',\varphi')$ and $\omega \in H^k_{cl}(M)$ then $-\int_C \varphi^*\omega + \int_{C'} \varphi'^*\omega = \int_B \Phi^* d\omega = 0$

So, have a pairing $\int : H^k_{dR}(M) \times \Omega^{SO}_k(M) \to \mathbb{R}$

Can also do the same replacing $\Omega^{SO}_k(M)$ with $H_k(M,\mathbb{R})$ (simplicial homology)

and then

**Thm (de Rham)** $\int : H^k_{dR}(M) \times H_k(M,\mathbb{R}) \longrightarrow \mathbb{R}$

is a nondegenerate bilinear pairing.

In particular, $H^k_{dR}(M) \simeq H^k(M,\mathbb{R})$.

(So $H^k_{dR}(M)$ is a topological invt — doesn't depend on smooth structure!

cf. Donaldson...)