More on orientations

1. $M, N$ oriented $\Rightarrow M \times N$ oriented
   
   $M \times N = (-1)^{\dim M \cdot \dim N} N \times M$.

2. $f: M \to N$, $Q \subset N$, $f \upharpoonright Q \quad P = f^{-1}(Q)$
   
   $df: NP \to f^*NQ$
   
   if $Q, N$ oriented then $NQ$ oriented, so this $NP$ oriented.
   
   if also $M$ oriented then get $P$ oriented.
   
   $NP = T_M/T_P$

Example

$S^2 = f^{-1}(1)$ for $f: A^3 \to \mathbb{R}$, $A^3, \mathbb{R}$ induce orientation in $S^2$.

Example

$\begin{array}{c}
\begin{array}{c}
\text{P} \\
\end{array} \\
\begin{array}{c}
\text{f} \\
\end{array} \\
\begin{array}{c}
\text{Q} \\
\end{array}
\end{array}$

Can view two ways: $\partial (f^{-1}(Q)) = (\partial f)^{-1}(Q)$

Prop

$\partial (f^{-1}(Q)) = (-1)^{\text{codim}(Q \subset N)} Q^c f^{-1}(Q)$.

Proof Exercise.

[One special case: $p \in M \Rightarrow f(p) \in N$, here get $Cf^{-1}(Q) = \partial (f^{-1}(Q)) = \partial (Q)$]

Lemma

$M$ oriented $1$-manifold $\Rightarrow \sum_{p \in M} \varepsilon(p) = 0$

$\varepsilon(p) = \begin{cases} +1 & \text{p is oriented} \\ -1 & \text{p is reversed orientation} \end{cases}$

Proof

Use classification of $1$-manifolds.

Degree

Def/Prop

$f: M \to N$ connected oriented $\Rightarrow$ connected oriented

$\deg f = \sum_{p \in f^{-1}(q)} \varepsilon(p)$

[where $\varepsilon = \begin{cases} +1 & \text{p is positively oriented} \\ -1 & \text{p is negatively oriented} \end{cases}$, $q$ regular value]

This is well defined and if $f \circ g$ then $\deg f = \deg g$.

Proof

As formal degree, except: we use the fact that if $F: I \times M \to N$ transversal $q$

then setting $P = F^{-1}(q)$, $\begin{array}{c}
\begin{array}{c}
\text{P} \\
\end{array} \\
\begin{array}{c}
\text{f} \\
\end{array} \\
\begin{array}{c}
\text{Q} \\
\end{array}
\end{array}$

have $\sum_{p \in P} \varepsilon(p) = 0$.

But also $\partial P = (-1)^{\dim M} (\partial F)^{-1}(q)$

$\Rightarrow \sum_{p \in \partial P} \varepsilon(p) = (-1)^{\dim M} (\deg F - \deg F_0)$

(recall the induced orientation: $\partial (I \times M) = \{1\} \times M \cup \{0\} \times -M$)
Prop \ \ \ M \xrightarrow{f} N \xrightarrow{g} \mathbb{R} \ : \ \ \deg (g \circ f) = \deg f \deg g.

Pf \ \ \ Pick \ r \in \mathbb{R} \ \text{s.t.} \ \text{r is regular value of } g \ \text{and each } q \in g^{-1}(r) \ \text{is regular value of } f.

Then, use \ \quad (g \circ f)^{-1}(r) = \bigsqcup_{q \in g^{-1}(r)} f^{-1}(q) \ \ \text{with } \ \xi_p^f \ \text{for oriented on } p \ \text{induced by } h

so that \ \quad \deg (g \circ f) = \sum_{p \in (g \circ f)^{-1}(r)} \epsilon_p^g \sum_{q \in f^{-1}(r)} \epsilon_p^f = \sum_{q \in f^{-1}(r)} \sum_{p \in g^{-1}(q)} \epsilon_q^g \epsilon_p^f

= \sum_{q \in f^{-1}(r)} \sum_{p \in g^{-1}(q)} \epsilon_p^f \epsilon_q^g = \sum_{q \in f^{-1}(r)} \epsilon_q^f \deg g = (\deg f)(\deg g)

Prop \ \quad \alpha: S^n \rightarrow S^n \ \text{antipodal map has } \deg (\alpha) = (-1)^{n+1}.

(x_1^0, \ldots, x^n) \mapsto (-x_1^0, \ldots, -x^n)

Pf \ \ \ \alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n \ \text{linear, induced orientation on } S^n. \ \alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n \ \text{given by same formula, } \alpha \circ \omega = \omega \circ \alpha.

Pick \ p = (1, 0, \ldots, 0) \in S^n. \ \text{Then orientation at } p \ \text{is given by } \omega_p = \epsilon^*(dx^1 \wedge \ldots \wedge dx^n) \ \text{since } dx^1 \ \text{is the oriented normal at } p \ \text{and } dx^1 \wedge \ldots \wedge dx^n \ \text{is the oriented normal on } \mathbb{R}^n.

At \ \alpha(p) \ \text{orientation is } \omega_{\alpha(p)} = -\epsilon^*(dx^1 \wedge \ldots \wedge dx^n).

And \ \epsilon^* \alpha^*(dx^1 \wedge \ldots \wedge dx^n) = \epsilon^* (\alpha^*(dx^1 \wedge \ldots \wedge dx^n))

= (-1)^{n+1} (dx^1 \wedge \ldots \wedge dx^n), \ \text{i.e. } \epsilon^* \omega_{\alpha(p)} = (-1)^{n+1} \omega_p.

Cor \ \quad S^n \ \text{admits nowhere-vanishing vector field } \iff n \ \text{is odd.}

Pf \ \ \ \text{For } n \ \text{odd, take the vector field } \xi(x) = (-x_1^0, x_2, \ldots, x^n, x_i^{-1}) \ \in T_x S^n = \{ \xi: \xi \cdot x = 0 \} \subset T_x \mathbb{R}^n.

For \ n \ \text{even, suppose we had such a vector field; then WLOG we may assume } \| \xi \| = 1 \ \forall x \in S^n;

then define map \ \quad F: \{0, \pi\} \times S^n \rightarrow S^n

(t, x) \mapsto (\cos t) x + (\sin t) \xi(x)

This gives a homotopy between \ F_0 = \text{identity and } F_\pi = \text{antipodal}

\deg F_0 = 1 \ \ \ deg F_\pi = -1

Prop \ \deg \xi = \deg f \ \text{mod } 2

Rk \ \ \ \text{Still, } \deg \xi \ \text{is useful when we have to deal with non-orientable manifolds.}
Def Prop \( f: M \to N \) \( M, Q \subset N \) of complementary dim \( M \) compact oriented \( N \) oriented 
\[ I(f, Q) = \sum_{p \in f^{-1}(Q)} \epsilon(p) \] when \( f \) is a fold \( f \neq Q \).

Pf well defined: again as before, now using our lemma above.

Notation If \( M \cap Q \) as above and \( M \subset N \), \( f: M \to N \) is inclusion, then write \( I(M, Q) = I(f, Q) \), ("oriented intersection number")

Ex 
\[
\begin{align*}
I(M, Q) &= +1 \\
I(Q, M) &= -1
\end{align*}
\]

Ex 
\[
\begin{align*}
P &= f^{-1}(Q) \text{ gets orientation from:} \\
\det N_{f^{-1}(Q)} \det TQ^T &\to \det TN \\
&\det Q \to \det \beta \\
&\beta &\to \det \beta
\end{align*}
\]

Ex \( M = N \), \( Q \) is oriented \( 0 \)-mfld: 
\[ I(M, Q) = \sum_{p \in Q} \epsilon(q). \]

Prop \[ I(M, Q) = (-1)^{(\dim M)(\dim Q)} I(Q, M) \] if both \( M, Q \subset N \) compact.

Pf Exercise, like one you already did for \( I_2 \) (and \( Z \) intersection).

Cor \( TRP^2 \) is not orientable.

Pf \( TRP^2 \) has a submanifold \( M \subset TRP^2 \) with transverse perturbation \( M' \) s.t. \( M \cap M' = pt. \)
So if \( TRP^2 \) were oriented we would have \( I(M, M) = \pm 1. \) But \( I(M, M) = (-1)^{\dim M} I(M, M) \)
so \( I(M, M) = 0. \)

Def \( M \) compact oriented smooth mfld: \( 0 \) diagonal \( \Delta_M = \{(p, p) : p \in M\} \subset M \times M \)
\[ \Delta_M \text{ is oriented since } \Delta_M \sim M \text{ canonically.} \]

(2) Euler characteristic \[ \chi(M) = I(\Delta_M, \Delta_M) \]

Ex \( M = S^1 \)
\[
\begin{align*}
\chi(S^1) &= I(\Delta_M, \Delta_M) = 0 \\
\end{align*}
\]

Ex \( M = pt \) : \( + \)
\[ \chi(pt) = I(pt, pt) = 1 \]
\[ M \times M = pt \\
\Delta_M = pt \]

Rk This def \( \chi \) can be extended to case where \( M \) is not oriented: just use a local orientation of \( M \) near each intersection point, note the sign \( \epsilon(p) \) doesn't depend on the local orientation.
Prop \quad \dim M \text{ odd} \Rightarrow \chi(M) = 0.

Pf \quad I(\Delta^m, \Delta^m) = (-1)^{(\dim M)^2} I(\Delta^m, \Delta^m) \Rightarrow I(\Delta^m, \Delta^m) = 0.