Lefschetz theory

Def $f: M \to M$, M compact oriented, $\Gamma_f = \{(p, f(p)) : p \in M\} \subset M \times M$:

$$L(f) = I(\Delta_M, \Gamma_f) \qquad (\text{"Lefschetz number"})$$

Some kind of measurement of the fixed points of f — if they're isolated, counts them with signs.

Ex $L(1_M) = \chi(M)$

Prop 1) If $f \sim g$ then $L(f) = L(g)$.
2) If $L(f) \neq 0$ then f has a fixed point.

Pf Easy.

Def ① $f: M \to M$, M compact oriented: $p \in M$ is <u>Lefschetz fixed point</u> of f if $\Gamma_f \pitchfork \Delta_M$ at (p,p)
② " " : f is <u>Lefschetz</u> if all its fixed pts are Lefschetz, ie $\Gamma_f \pitchfork \Delta_M$.

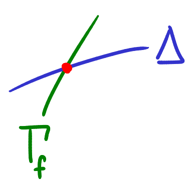
Prop $f: M \to M$, M compact oriented: $\exists\, g \sim f$ s.t. g is Lefschetz.

Pf Can find $F: M \times S \to M$ s.t. $F(\cdot, 0) = f$ and $F(p, \cdot)$ submersion $\forall p$ (we showed this earlier).

This $\Rightarrow$
$$G: M \times S \to M \times M \qquad \text{is also submersion.}$$
$$(p, \sigma) \mapsto (p, F(p, \sigma))$$

Then apply transversality thm $\Rightarrow$ for almost every $\sigma \in S$, $[p \mapsto (p, F(p,\sigma))] \pitchfork \Delta_M$. ∎

What Lefschetz fixed pts are like:

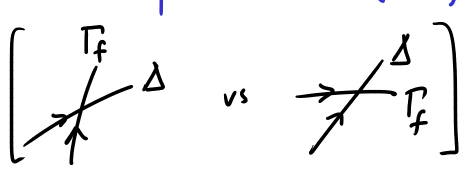


$$T_{(p,p)}\Gamma_f \cap T_{(p,p)}\Delta = \{0\}$$
$$\{(\xi, df_p(\xi))\} \qquad \{(\xi, \xi)\}$$

ie df_p has no eigenvalue $+1$
(infinitesimal analogue of saying p is <u>isolated</u>)

Def p Lefschetz fixed pt of f: $L_p(f)$ ("Lefschetz #") is contrib^{n} of p to $L(f)$. $(= \pm 1)$

Prop $L_p(f)$ is the sign of $\det(df_p - 1)$.



Pf Take $\{e_1, \ldots, e_m\}$ +ve oriented basis for T_pM.
Lefschetz # is the sign of the basis
$$\{\underbrace{(e_1,e_1), (e_2,e_2), \ldots, (e_m,e_m)}_{T_{(p,p)}\Delta}, \underbrace{(e_1, df_p e_1), \ldots, (e_m, df_p e_m)}_{T_{(p,p)}\Gamma_f}\}$$
relative to orientation of $M \times M$.

Now make "row operations" on this basis:

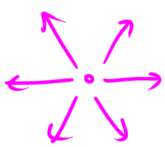
$\sim \{(e_1,e_1),(e_2,e_2),\ldots,(e_m,e_m),\ (0,(df_p-1)e_1),\ldots,(0,(df_p-1)e_m)\}$

$\sim \{(e_1,0),(e_2,0),\ldots,(e_m,0),\ (0,(df_p-1)e_1),\ldots,(0,(df_p-1)e_m)\}$

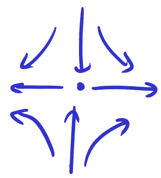
which differs from +ve basis for $T_{(p,p)}M\times M$ by matrix $\begin{pmatrix} \mathbb{1} & 0 \\ 0 & df_p-\mathbb{1}\end{pmatrix}$ ∎

Ex case $m=2$, if $df_p \sim \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$: local behavior

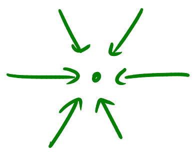
$\lambda_1,\lambda_2>1$:
$L_p(f)=1$



$\lambda_1>1,\ \lambda_2<1$:
$L_p(f)=-1$

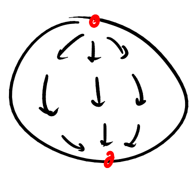


$\lambda_1,\lambda_2<1$:
$L_p(f)=1$



Cor $\chi(S^2)=2$.

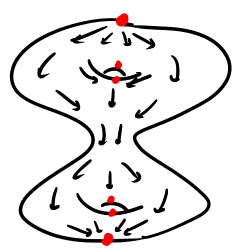
Pf Let $f: S^2\to S^2$ be flow along a vector field pointing "south", vanishing at poles.



2 fixed pts with $L_p(f)=+1$.
So $L(f)=2$; and $f\sim\mathbb{1}$.
Thus $L(\mathbb{1})=2$. ∎

Cor $\chi(\Sigma)=2-2g$ for Σ of genus g.

Pf Sketch Flow down along the surface:
$f:\Sigma\to\Sigma$ has 2 fixed pts with $L_p(f)=+1$, $2g$ with $L_p(f)=-1$.



∎

How about maps which are not Lefschetz? Want to describe directly the contrib. from degenerate fixed pts.
We can "split" them:

Prop p fixed pt of $f: M\to M$, U nbhd of p cont. no other fixed pt
$\Rightarrow \exists\, g: M\to M$, $f\sim g$, $f=g$ outside compact $K\subset U$, $g|_U$ Lefschetz.

Pf Say $U\subset \mathbb{A}^m$ and $p=0$. Take $\rho: \mathbb{A}^m\to[0,1]$ smooth, $\rho=1$ on $V\subset U$ open,

$\rho=0$ off $K\subset U$ compact. For $\sigma\in\mathbb{A}^m$ let $g(x)=f(x)+\rho(x)\sigma$.

For $\|\sigma\|$ small enough, g has no fixed pts on $U\setminus V$ (show).

Choose σ regular value for $x\mapsto f(x)-x$ (Sard).

All fixed pts of g have $\sigma=f(x)-x$, thus df_x-1 is $\simeq$, ie they are Lefschetz.

Then transferring to a general M is straightforward. $\blacksquare$

Now, how to detect the local contribution from an isolated fixed pt. <u>without</u> splitting it?

$[x^2-y^2, xy]$ $\qquad$ $[x^2-y^2-1, xy]$

Look at <u>winding</u> of f near the fixed point:

<u>Def/Prop</u> For $f:\mathbb{A}^m\to\mathbb{A}^m$ s.t. $p=0$ is isolated fixed point, fix a ball $B_\varepsilon(0)$ containing no <u>other</u> fixed point, then define Lefschetz #,

$$L_0(f)=\deg\left(\begin{array}{c} \varphi_\varepsilon:\ \underset{\cong\, S^{m-1}}{\partial B_\varepsilon(0)}\longrightarrow S^{m-1} \\ x\longmapsto \dfrac{f(x)-x}{\|f(x)-x\|}\end{array}\right)$$

If p is Lefschetz fixed point, this agrees with our previous def of $L_p(f)$.

Pf Well defined: changing ε changes the map φ_ε by a homotopy.

For p Lefschetz, make a homotopy $f\sim g$ on B_ε, where $g(x)=[df_p(0)](x)$ (use Taylor thm). Thus we want the degree of

$$x\longmapsto\frac{(df_0-1)x}{\|(df_0-1)x\|}$$

then, using the fact that $GL_+(m)$ is <u>connected</u> [Exercise], can homotope <u>this</u> map to $\quad x\longmapsto\dfrac{x}{\|x\|}\quad$ if df_0-1 <u>preserves</u> orientation

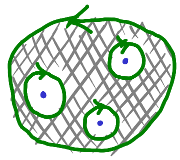
or $\quad x\longmapsto\dfrac{Rx}{\|x\|}\quad Rx=(-x^1,\dots,x^m)\quad$ if df_0-1 <u>reverses</u> orientation

and note these maps have degree ±1 as needed.

Prop $f: \mathbb{A}^m \to \mathbb{A}^m$ isolated fixed point at 0, $B = B_\varepsilon(0)$, $\overline{B}$ has no other fixed point of f, $g: \mathbb{A}^m \to \mathbb{A}^m$ has $g = f$ outside $K \subset B$ compact and only Lefschetz fixed pts in B: then

$$L_0(f) = \sum_{\substack{p \in B \\ \text{fixed} \\ \text{for } g}} L_p(g).$$

Pf $L_0(f) = $ degree of $x \mapsto \dfrac{f(x)-x}{\|f(x)-x\|}$ on $\partial\overline{B}$ $=$ degree of $G: x \mapsto \dfrac{g(x)-x}{\|g(x)-x\|}$ on $\partial\overline{B}$

And $\partial C = \partial\overline{B} - \bigcup_i \partial\overline{B}_i$,  G extends to C, so $\deg(G)$ on ∂C is 0,

so $L_0(f) = $ degree of G on $\bigcup_i \partial\overline{B}_i = \sum_{\substack{p \in B \\ \text{fixed} \\ \text{for } g}} L_p(g)$. ∎

Def/Prop For $f: M \to M$ s.t. p is isolated fixed point, $L_p(f) = L_p(x \circ f \circ x^{-1})$ for (U, x) chart at p

Pf Check well defined: if p is Lefschetz then

$$\begin{aligned} L_p(x \circ f \circ x^{-1}) &= \operatorname{sgn}\det\left(d(x \circ f \circ x^{-1}) - I\right) \\ &= \operatorname{sgn}\det\left(d(y \circ x^{-1}) \circ \left(d(x \circ f \circ x^{-1}) - I\right) \circ d(x \circ y^{-1})\right) \\ &= \operatorname{sgn}\det\left(d(y \circ f \circ y^{-1}) - I\right) \end{aligned}$$

If p is not Lefschetz then break it into Lefschetz fixed pts, use the last Prop to see that $L_p(f)$ is a sum of their indiv. Lefschetz #'s, which are indep of chart by the above ∎

Prop $f: M \to M$ smooth, finite # fixed pts: $L(f) = \sum_{f(p)=p} L_p(f)$

Pf Perturb f around each fixed pt to g Lefschetz, then $L(f) = \sum_{g(p)=p} L_p(g)$ and by the above Prop this is also $\sum_{f(p)=p} L_p(f)$. ∎