Lefschetz theory

\textbf{Def} \quad f: M \to M, M \text{ compact oriented}, T_f^p = \{(p, f(p)): p \in M\} \subset M \times M:

\[ L(f) = \int (\Delta, T_f) \quad (\text{"Lefschetz number"}) \]

Some kind of measurement of the fixed points of \( f \) — if they're isolated, counts them with signs.

\textbf{Ex} \quad L(1_M) = X(M)

\textbf{Prop} \quad 1) If \( f \sim g \) then \( L(f) = L(g) \).

2) If \( L(f) \neq 0 \) then \( f \) has a fixed point.

\textbf{Pf} \quad \text{Easy.}

\textbf{Def} \quad 1) f: M \to M, M \text{ compact oriented}: p \in M \text{ is Lefschetz fixed point of } f \text{ if } T_f^p \pitchfork \Delta_M \text{ at } (p, p).

2) \( f \) \text{ is Lefschetz if all its fixed pts are Lefschetz, i.e., } T_f^p \pitchfork \Delta_M.

\textbf{Prop} \quad f: M \to M, M \text{ compact oriented}: \exists g \sim f \text{ s.t. } g \text{ is Lefschetz.}

\textbf{Pf} \quad \text{Consider } F: M \times S \to M \text{ s.t. } F(\cdot, 0) = f \text{ and } F(p, \cdot) \text{ submersion } \forall p \text{ (we showed this earlier).}

\[ \text{This } \Rightarrow G: M \times S \to M \times M \text{ is also submersion.} \]

\[ (p, s) \mapsto (p, F(p, s)) \]

Then apply transversality then \( \Rightarrow \) for almost every \( s \in S \), \( [p \mapsto (p, F(p, s))] \pitchfork \Delta_M. \]

\textbf{What Lefschetz fixed pt are like:}

\[ \Delta \]

\[ T_f^p \cap T_f^p \Delta = \{0\} \]

\[ \{ (\xi, df_p(\xi)) \} \quad \{ (\xi, \xi) \} \]

\text{Hence } df_p \text{ has no eigenvalue } +1

\text{(infinitesimal analogue of saying } p \text{ is isolated)}

\textbf{Def} \quad p \text{ Lefschetz fixed pt of } f: L_p(f) \text{ ("lefschetz#") is critical of } p \text{ to } L(f). \quad (= \pm 1)

\textbf{Prop} \quad L_p(f) \text{ is the sign of } \det(df_f-1).

\textbf{Pf} \quad \text{Take } \{e_1, \ldots, e_m\} \text{ t.v. oriented basis for } T_p M.

\text{Lefschetz # is the sign of the basis}

\[ \{ (e_1, e_1), (e_2, e_2), \ldots, (e_m, e_m), (e_1, df_p e_1), \ldots, (e_m, df_p e_m) \} \]

\text{relative to orientation of } M \times M.
Now make "row operations" on this basis:
\[
\sim \{ (e_1, e_2), \ldots, (e_m, e_m), (0, (dp-1)e_1), \ldots, (0, (dp-1)e_m) \}
\]
\[
\sim \{ (e_1', e_2'), \ldots, (e_m', e_m'), (0, (dp-1)e_1'), \ldots, (0, (dp-1)e_m') \}
\]
which differs from the basis for \( T_{(p,p)} \mathbb{M}/\mathbb{M} \) by making
\[
\begin{pmatrix}
1 & 0 \\
0 & dp-1
\end{pmatrix}
\]

**Ex:** case \( m = 2 \), if \( dp \sim \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \):

- \( \lambda_1, \lambda_2 > 1 \):
  \( L_p(f) = 1 \)
- \( \lambda_1, \lambda_2 < 1 \):
  \( L_p(f) = 1 \)
- \( \lambda_1 > 1, \lambda_2 < 1 \):
  \( L_p(f) = -1 \)
- \( \lambda_1 < 1, \lambda_2 > 1 \):
  \( L_p(f) = 1 \)

**Cor:** \( \chi(S^2) = 2 \).

**Pf**
Let \( f: S^2 \to S^2 \) be flow along a vector field
pointing "south", vanishing at poles.
2 fixed pts with \( L_p(f) = +1 \).
So \( L(f) = 2 \); and \( f \sim 1 \).
Thus \( L(f) = 2 \).

**Cor:** \( \chi(\Sigma) = 2 - 2g \) for \( \Sigma \) of genus \( g \).

**Pf Sketch** Flow down along the surface:
\( f: \Sigma \to \Sigma \) has 2 fixed pts
with \( L_p(f) = +1 \), \( 2g \) with \( L_p(f) = -1 \).

How about maps which are not Lefschetz? Want to describe directly the control from degenerate fixed pts.
We can "split" them:

**Prop** p fixed pt of \( f: M \to M \), U nbhd of p cont. no other fixed pt
\( \Rightarrow \exists g: M \to M \), f \sim g, \( f \sim g \) outside compact \( K \subset U \), \( g|_U \) Lefschetz.

**Pf** Say \( U \subset \mathbb{R}^m \) and \( p = 0 \). Take \( \rho: \mathbb{A}^m \to [0,1] \) smooth, \( \rho = 1 \) on \( V \subset U \) open,
For $\sigma \in \mathbb{A}^m$ let $g(x) = f(x) + \rho(x)\sigma$. For $\|x\|$ small enough, $g$ has no fixed pts on $U \setminus V$ (show).

Choose a regular value $x \mapsto f(b) - x$ (Sard).

All fixed pts of $g$ have $\sigma = f(x) - x$, thus $df_x - 1$ is $= 0$, i.e. they are Lefschetz. Then transfer to a general $M$ is straightforward.

Now, how to detect the local contribution from an isolated fixed pt. without splitting it?

\[ \begin{array}{c}
\begin{array}{c}
\circ \\
[xy, x^2-y^2, xy]
\end{array}
\end{array} \quad \mapsto \quad \begin{array}{c}
\begin{array}{c}
\circ \\
[xy, x^2-y^2-1, xy]
\end{array}
\end{array} \]

Look at winding of $f$ near the fixed point:

**Def/Prop** For $f : \mathbb{A}^m \to \mathbb{A}^m$ s.t. $p=0$ is isolated fixed point, fix a ball $B_\varepsilon(0)$ containing no other fixed point, then define Lefschetz $L_0:f$,

\[ L_0(f) = \text{deg} \left( \varphi_\varepsilon : \partial B_\varepsilon(0) \to S^{m-1} \right) \]

\[ \begin{array}{c}
\varphi_\varepsilon : \\
\begin{array}{c}
\bigoplus \bigoplus \\
\begin{array}{c}
\text{S}^1 \\
\int f(x) - x \\
\int \|f(b) - x\|
\end{array}
\end{array}
\end{array} \]

If $p$ is Lefschetz fixed point, this agrees with our previous def of $L_0(f)$.

**Pf** Well defined: changing $\varepsilon$ changes the map $\varphi_\varepsilon$ by a homotopy. For $p$ Lefschetz, make a homotopy $f \to g$ on $B_\varepsilon$, where $g(x) = (df_p(0))(x)$ (use Taylor thm). Thus we want the degree of $x \mapsto \frac{(df_p(0))x}{\|df_p(0)x\|}$

then, using the fact that $GL_\varepsilon$ is connected [Exercise], can

homotopy this map to $x \mapsto \frac{x}{\|x\|}$ if $df_p - 1$ preserves orientation

\[ \text{or } x \mapsto \frac{Rx}{\|x\|} \quad \text{Rx} = (-x', \ldots, x^m) \quad \text{if } df_p - 1 \text{ reverses orientation} \]

and note these maps have degree $\pm 1$ as needed.
Prop: f: \( \mathbb{R}^m \rightarrow \mathbb{R}^m \) isolated fixed point at 0, \( B = B_\varepsilon(0) \), \( \overline{B} \) has no other fixed point of \( f \)

Then

\[
L_0(f) = \sum_{p \in \overline{B} \setminus \{0\}} L_p(g).
\]

Proof:

\[ L_0(f) = \text{degree of } x \mapsto \frac{f(x) - x}{\|f(x) - x\|} \text{ on } \overline{B} = \text{degree of } G: x \mapsto \frac{g(x) - x}{\|g(x) - x\|} \text{ on } \overline{B} \]

And \( \mathcal{C} = \overline{B} - \bigcup_i \overline{B}_i \), \( G \) extends to \( C \), so \( \deg(G) \) on \( \mathcal{C} \) is 0,

so

\[ L_0(f) = \text{degree of } G \text{ on } \bigcup_i \overline{B}_i = \sum_{p \in \bigcup_i \overline{B}_i \setminus \{0\}} L_p(g). \]

Def/Prop: For \( f: M \rightarrow M \) s.t. \( p \) is isolated fixed point, \( L_p(f) = L_p(x \cdot f \cdot x^{-1}) \) for \((x,y)\) chart at \( p \)

Proof:

Check well defined: if \( p \) is Lefschetz then \( L_p(x \cdot f \cdot x^{-1}) = \text{sgn } \det(d(x \cdot f \cdot x^{-1}) - 1) \)

\[ = \text{sgn } \det(d(y \cdot x^{-1}) \cdot (d(x \cdot f \cdot x^{-1}) - 1) \cdot d(x \cdot y^{-1}) \]

\[ = \text{sgn } \det(d(y \cdot f \cdot y^{-1}) - 1) \]

If \( p \) is not Lefschetz then break it into Lefschetz fixed pts, use the last Prop to see that \( L_p(f) \) is a sum of their indep. Lefschetz #s, which are indep of chart by the above.

Prop: \( f: M \rightarrow M \) smooth, finite # fixed pts:

\[ L(f) = \sum_{p \in \text{fixed pt}} L_p(f) \]

Proof:

Periods f around each fixed pt do \( g \) Lefschetz, then \( L(f) = \sum_{p \in \text{fixed pt}} L_p(g) \) and by the above Prop this is also \( \sum_{p \in \text{fixed pt}} L_p(f) \).