Def/Prop \( \xi \in \chi(M) \) with isolated zero at \( x = 0 \), no other zeros on ball \( B \): \( \xi \) gives a map \( \mathbb{R}^n \to S^{n-1} \).

Let \( \text{ind}_0 \xi \) be the degree of this map.

This agrees with the Lefschetz # \( L_p(f) \) where \( f_t \) is the flow of \( \xi \) for a time \( t \).

\[ \begin{array}{c}
\text{Ex} \\
\begin{array}{c}
\text{Ind}_p \xi = 1 \\
\text{Ind}_p \xi = -1
\end{array}
\end{array} \]

PF Sketch: For small enough \( t \), \( f_t \) has no fixed points in \( B \) except for \( x = 0 \). (This needs proof; see e.g., http://math.stackexchange.com/questions/48074).

Then, \( \frac{f_t(x)}{\|f_t(x)\|} = \frac{t \xi(x) + \xi^2 r(x,t)}{\|t \xi(x) + \xi^2 r(x,t)\|} \) by Taylor's thm, \( r(x,t) \) smooth, and the maps on RHS are all homotopic.

\( \Rightarrow \) degree is indy of \( t \). At \( t = 0 \), get \( \frac{\xi(x)}{\|\xi(x)\|} \).

Def/Prop \( \xi \in \chi(M) \) with isolated zero at \( p \in M \): take a chart \( (U,x) \) containing \( p \),
then \( \text{dx} \circ \xi \circ x^{-1} \bigg|_{\{x\}} \) gives a map \( S^{n-1} \to S^{n-1} \).

Let \( \text{ind}_p \xi \) be the degree of this map.

This agrees with the Lefschetz # \( L_p(f) \) where \( f_t \) is the flow of \( \xi \) for a sufficiently small time \( t \).

Cor (Poincaré–Hopf) \( M \) compact, \( \xi \in \chi(M) \), all zeros of \( \xi \) isolated:
\[ \chi(M) = \sum_{p \in \text{zeros} \xi} \text{ind}_p(\xi). \]

Rk
1. \( L(f) \) can be defined even for \( M \) not compact.
2. A wonder-ful! Thm which we want prove says:
   so e.g. \( \chi(M) = \sum_{i=0}^{m} (-1)^i \dim H^i_{\text{dR}}(M) \)
\[ L(f) = \sum_{i=0}^{m} (-1)^i \text{Tr}[f^*: H^i_{\text{dR}}(M) \to H^i_{\text{dR}}(M)]. \]
Say $M$ is a triangulated 2-manifold. Then one can find a vector field $\xi \in \mathfrak{X}(M)$ which looks in each face like the picture:

\[
\begin{align*}
1 \text{ zero in each face, index } +1 \\
1 \text{ zero in each edge, index } -1 \\
1 \text{ zero at each vertex, index } +1
\end{align*}
\]

\[\Rightarrow \mathfrak{X}(M) = \# \text{faces} - \# \text{edges} + \# \text{vertices} \quad \text{(And similarly for n-dimensional manifolds!)}\]

More on vector fields

**Prop.** Suppose $\xi_1 \in \mathfrak{X}(M)$ and $\xi_1(\mathbf{p}) \neq 0$.

Then, $\exists$ a chart $(U,x)$ with $p \in U$ s.t. $\xi_1 = \frac{2}{\partial x^i}$

**Pf.** We can reduce to case $M = \mathbb{R}^m$ with coord $x^1, \ldots, x^m$ and assume $\xi_1(0) = \frac{2}{\partial x^i}$. Idea: unique integral curve of $\xi_1$ through any point $(0, t^1, \ldots, t^m)$. For $q$ on this curve take $(x^1, \ldots, x^m) = (c, t^1, \ldots, t^m)$ where $c = \text{time to reach } q$ along the flow.

ie: let $\phi_t$ be flow generated by $\xi_1$, then set $X(x^1, \ldots, x^m) = \phi_t(x^1, \ldots, x^m)$

Then $dX(\frac{\partial}{\partial x^i}) = \xi_1$ (def. of "flow") and $dX_0(\frac{\partial}{\partial x^i}) = \frac{2}{\partial x^i}$ for $i \neq 1$. Thus $dX_0 = Id$, i.e. $X$ is local diffeo around $0$, so $X^{-1}$ gives a chart around $0$ with the desired property.

But what about two vector fields $\xi_1, \xi_2$?

An obstruction to realizing $\xi_1 = \frac{\partial}{\partial x^1}$ and $\xi_2 = \frac{\partial}{\partial x^2}$:

**Def/Prop.** For $\xi_1, \xi_2 \in \mathfrak{X}(M)$, $[\xi_1, \xi_2] \in \mathfrak{X}(M)$ is determined by $[\xi_1, \xi_2]f = \xi_1(\xi_2f - \xi_2(\xi_1f$).

**Pf.** On $M = \mathbb{R}^m$:

$\xi_1 = g^i(\frac{\partial}{\partial x^i}), \quad \xi_2 = h^j(\frac{\partial}{\partial x^j})$

$\xi_1(\xi_2f - \xi_2(\xi_1f) = (g^i(\frac{\partial}{\partial x^i})(h^j(\frac{\partial}{\partial x^j})))f - (h^j(\frac{\partial}{\partial x^j}))(g^i(\frac{\partial}{\partial x^i}))f$

$= g^ih^j\frac{\partial f}{\partial x^i} + g^i h^j \frac{\partial f}{\partial x^j} = h^j \frac{\partial}{\partial x^j} f - h^j \frac{\partial}{\partial x^j} f$

$= [g^i \frac{\partial}{\partial x^i} - h^j \frac{\partial}{\partial x^j}]f$

So if we set $[\xi_1, \xi_2] = [g^i \frac{\partial}{\partial x^i} - h^j \frac{\partial}{\partial x^j}].$ Then we get $(\xi_1(\xi_2 - \xi_2(\xi_1)f = [\xi_1, \xi_2]f$

This formula is enough to determine what happens on general $M$. But let's spell it out carefully.

Then for general $M$, with $\xi_1, \xi_2 \in \mathfrak{X}(M)$, pick a chart $(x, U)$ and define $[\xi_1, \xi_2] \in \mathfrak{X}(U)$ by requiring

$\left( d\xi([\xi_1, \xi_2])f \right) = \left( d\xi(\xi_1))d\xi(\xi_2)\right)f - \left( d\xi(\xi_2))d\xi(\xi_1)\right)f$

$\Rightarrow f \in C^\infty(U)$

To check independence of chart: take $(y, V)$ another chart. Let $\phi = xy^{-1}$.

\[x(U) = \phi \circ y(U)\]
Then \( dy([x, y]) \cdot (f \circ \phi) = \left[ df^i \left( dx([x, y]) \right) \right] \phi^* f = \phi^* (dx([x, y]) f) \)
\( \nabla (y(v)) C(y(v)) \)

and similarly

\( dy([x, y]) f - dy([x, x]) f = \phi^* \left[ (dy([x, y]), dy([x, x])) f \right] \)

Thus, get \( (dy([x, y])) f = (dy([x, x])) f - (dy([x, y])) f \)

So \([x, y] \) is indeed well defined by this equation.

\begin{itemize}
  \item [1)] \([x, y] = -[y, x]\)
  \item [2)] \([y, f \cdot z] = (y, f) \cdot z + f \cdot [y, z]\)
  \item [3)] \([x, f] = f \cdot x - f ,\ f\ f = 0.\)
\end{itemize}

This means that \([y, z] \neq 0\) there cannot be a coordinate with \(y = x, z = y\).

For \(w \in \Omega^2(M), dw \in \Omega^3(M)\) is given by the formula

\[
dw : \Omega(M) \times \Omega(M) \to \Omega^2(M) \]

\[
dw([x, y]) = [x, [y, x]] - [y, (x, x)] - \omega([x, y]) \]

\([x, y] \in \Omega(M)\)

\(\omega([x, y]) = \omega([x, y]) - \omega((x, y)) \)

**Proof**

First check this formula actually defines some 2-form. This means checking it's

1) **antisymmetric in \([x, y] \to [y, x]\)**

2) **linear over \(\Omega^0(M)\):**

\[
f \cdot [x, y] = [f \cdot x, y] - [x, f \cdot y] - \omega([x, y])
\]

**Fubini**

\([x, y] \in \Omega(M):\) \(\omega([x, y]) = 0 \forall i, j = 1, \ldots, k\) then \(A \in M, C\) chart \((x, U)\) at \(p\) such that

\(\omega_{ij} \in \Omega\)

**Shell**

Some idea is above for \(l = 1\). But we need to know that the flows generated by \(\omega_{ij}\) commute!
Def 1: \( \mathbf{M} \to \mathbf{N}, \ X \in \mathfrak{X}(\mathbf{M}), \ Y \in \mathfrak{X}(\mathbf{N}) \): say \( X \) and \( Y \) are \textit{f-related} if \( \forall p \in \mathbf{X}, \ Y(\phi(p)) = d\phi_p(X(p)). \)

2. If \( \phi: \mathbf{M} \to \mathbf{N} \) differs, any \( X \in \mathfrak{X}(\mathbf{M}) \) is \textit{f-related} to unique set \( f \in \mathfrak{X}(\mathbf{N}) \).

Prop If \( X, Y \) are \textit{f-related} and \( X', Y' \) are \textit{f-related} then \( [X, X'] \) and \( [Y, Y'] \) are \textit{f-related}.

Pf \( [Y, Y'] g \cdot f = (YY' g) f - (YY' g) f = X(Y' g f) - X'(Y g f) = X X' (g f) - X' X (g f) = [X, X'] (g f). \)

Def 1. A \textbf{distribution} \( \mathcal{D} \) on \( \mathbf{M} \) is a subbundle of \( \mathcal{T}\mathbf{M} \).

2. \( \mathcal{D} \) is \textit{involutory} if \( \forall X, Y \in \mathfrak{T} \mathcal{D} \subset \tilde{\mathfrak{X}}(\mathbf{TM}) \ [X, Y] \in \mathfrak{T} \mathcal{D}. \)

3. \( \mathcal{D} \) is \textit{integrable} if \( \mathbf{M} \) can be covered by charts \( (x, U) \) s.t. \( x(U) \cap \{x^i = a^j, x^k = a^k\} \) is a submanifold \( \mathcal{N}^a \), with \( T_p \mathcal{N}^a = \mathcal{D}_p, \forall a. \)

Cor \( \mathcal{D} \) is involutory \( \iff \) \( \mathcal{D} \) is integrable.

Pf (\( \iff \)) is easy.

(\( \implies \)) enough to do it locally in \( \mathbf{M} \subset \mathbb{R}^m \) open. Can assume \( \mathcal{D} = \langle \partial_1, \ldots, \partial_k \rangle \). Then consider the projection map \( \pi: \mathbf{M} \to \mathbb{R}^k \); on some nbhd \( U \) of \( 0 \), \( d\pi_p: \mathcal{D}_p \to \mathbb{R}^k \). Then \( \frac{\partial}{\partial p^i} = d\pi_p \left( \frac{\partial}{\partial x^i} \right) \) are vector fields on \( M \), and \( [\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] \in \mathfrak{T} \mathcal{D} \) is \textit{f-related} to \( \left[ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0 \), thus \( [\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0 \). Then use the Frobenius theorem.

To understand why flows commute, and complete pf of Frobenius then:

Def \( X \in \mathfrak{X}(\mathbf{M}) \): \begin{enumerate}
\item \( \exists \xi \in \mathfrak{X}(\mathbf{M}): \mathcal{L}_X \xi \in \mathfrak{X}(\mathbf{M}) \) (Lie derivative) given by \( \mathcal{L}_X \xi(p) = \lim_{t \to 0} \frac{1}{t} \left[ \xi(p + t\phi_t) - \xi(p) \right] \)
\item \( \omega \in \Omega(\mathbf{M}) \): \( \mathcal{L}_X \omega \in \Omega(\mathbf{M}) \)
\end{enumerate}

Ex \( M = \mathbb{R}^2, X = \partial_x \): \( \mathcal{L}_X (x \, dx) = \lim_{t \to 0} \frac{1}{t} [(x + t) \, dx - x \, dx] = \frac{dx}{x^2} \), \( \mathcal{L}_X (x \, dy) = \lim_{t \to 0} \frac{1}{t} [(x + t) \, dy - x \, dy] = 0 \)

Prop \( X, \xi \in \mathfrak{X}(\mathbf{M}) \): \( \mathcal{L}_X \xi = [X, \xi] \)

Pf local coordinates: \( X = f^i \partial_i, \xi = g^j \partial_j \) \( p = 0 \)

\( \mathcal{L}_X \xi(0) = \lim_{t \to 0} \frac{1}{t} \left[ \xi_0(0) (x(t)) - \xi_0(t) \right] = \lim_{t \to 0} \frac{1}{t} \left[ g^i(0) \partial_i - t \frac{2 \partial}{2 x^i} g^i + \ldots - g^i (-t f^i(0) + \ldots) \partial_i \right] = \ldots \)

Lemma (removal): \( f: \mathbf{M} \to \mathbf{N}, \phi: \mathbf{I} \to \mathbf{M} \): \( \frac{d}{dt} (f(\phi(t))) = df_{\phi(t)} (\frac{d}{dt} \phi(t)) \)

Pf Chain rule.
Prop \( f : M \rightarrow N, X \in \mathfrak{X}(M), Y \in \mathfrak{X}(N) \) \( f \)-related, \( \phi_t \) flow of \( X \), \( \phi_{f(t)} \) flow of \( Y \): \( \phi_{f(t)} \circ f = f \circ \phi_t \).

Proof
They agree at \( t=0 \) and \( \frac{d}{dt} \phi_{f(t)}(p) = Y(f(\phi_t(p))), \quad \frac{d}{dt} (f(\phi_t(p))) = df_{\phi_t(p)}(X(\phi_t(p))) = Y(f(\phi_t(p))) \) in both sides obey the same first-order ODE.

Prop \( X, Y \in \mathfrak{X}(M), [X,Y]=0 \), \( \phi_t \) flow of \( X \), \( \phi_t \) flow of \( Y \): \( \phi_t \circ \phi_s = \phi_{t+s} \).

Proof
Set \( f = \phi_s \) in the above prop. So, we want to show \( X \) is \( \phi_s \)-related to itself, i.e. \( X = \phi_s X \).

Clearly true for \( s = 0 \), so consider
\[
\frac{d}{ds} \phi_s X(p) = \lim_{h \to 0} \frac{\phi_{s+h}X(p) - \phi_s X(p)}{h}
\]

\[
= \lim_{h \to 0} \phi_s \left( \frac{\phi_h X(p) - X(p)}{h} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{d(\phi_h X)}{d(p)} \right) \left( \phi_s \frac{(\phi_h X(p) - X(p))}{h} \right)
\]

\[
= \left. \left( \frac{d(\phi_s X)}{d(p)} \right) \right|_{h \to 0} \left[ \frac{d(\phi_s X)}{d(p)} \right] = 0 \quad \text{as desired.}
\]

Finally, this finishes pf of Frobenius theorem.