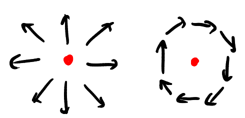
Def/Prop $\xi \in \mathfrak{X}(\mathbb{A}^n)$ with isolated zero at $x=0$, no other zeroes on ball B: $\frac{\xi}{\|\xi\|}$ gives a map $\underset{S^{n-1}}{\underset{\shortparallel}{\partial B}} \longrightarrow S^{n-1}$

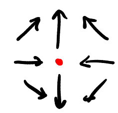
Let $\mathrm{ind}_0 \xi$ be the degree of this map.

This agrees with the Lefschetz # $L_0(f)$ where f_t is the flow of ξ for a time t.

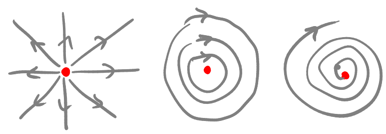
Ex

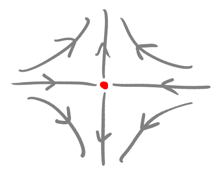


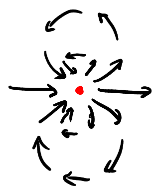
$\mathrm{ind}_p \xi = 1$



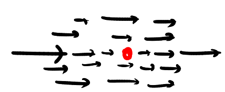
$\mathrm{ind}_p \xi = -1$



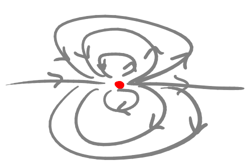


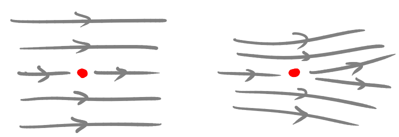


$\mathrm{ind}_p \xi = 2$



$\mathrm{ind}_p \xi = 0$





Pf Sketch For small enough t, f_t has no fixed points in B except for $x=0$. (This needs proof, see e.g. http://math.stackexchange.com/questions/48074)

Then, $\frac{f_t(x)}{\|f_t(x)\|} = \frac{t\xi(x) + t^2 r(x,t)}{\|t\xi(x) + t^2 r(x,t)\|}$ by Taylor's thm, $r(x,t)$ smooth, and the maps on RHS are all homotopic

$\Rightarrow$ degree is indep of t. At $t=0$, get $\frac{\xi(x)}{\|\xi(x)\|}$. ∎

Def/Prop $\xi \in \mathfrak{X}(M)$ with isolated zero at $p \in M$: take a chart (U,x) containing p,

then $dx \circ \frac{\xi}{\|\xi\|} \circ x^{-1} \Big|_{\{x : \|x - x(p)\| = \varepsilon\}}$ gives a map $S^{n-1} \longrightarrow S^{n-1}$

Let $\mathrm{ind}_p \xi$ be the degree of this map.

This agrees with the Lefschetz # $L_p(f)$ where f_t is the flow of ξ for a sufficiently small time t.

Cor (Poincaré-Hopf) M compact, $\xi \in \mathfrak{X}(M)$, all zeroes of ξ isolated:

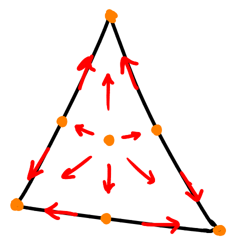
$$\chi(M) = \sum_{p \in \mathrm{zeroes}(\xi)} \mathrm{ind}_p(\xi).$$

Rk 1) $L(f)$ can be defined even for M not oriented.

2) A wonderful Thm which we won't prove says: $L(f) = \sum_{i=0}^{m} (-1)^i \mathrm{Tr}\left[f^* : H^i_{dR}(M) \to H^i_{dR}(M)\right]$

So e.g. $\chi(M) = \sum_{i=0}^{m} (-1)^i \dim H^i_{dR}(M)$

Triangulations



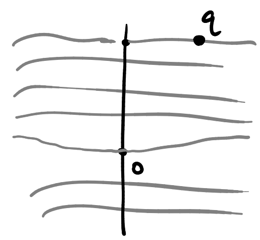
Say M is a triangulated 2-manifold. Then can find a vector field $\xi \in \mathfrak{X}(M)$ which looks in each face like the picture:

$$\left.\begin{array}{l} 1 \text{ zero in each face, index } +1 \\ 1 \text{ zero in each edge, index } -1 \\ 1 \text{ zero at each vertex, index } +1 \end{array}\right\} \Longrightarrow \chi(M) = \#\text{faces} - \#\text{edges} + \#\text{vertices}$$

(And similarly for n-dimensional manifolds!)

More on vector fields

Prop Suppose $\xi_1 \subset \mathfrak{X}(M)$ and $\xi_1(p) \neq 0$.
Then, $\exists$ chart (U, x) with $p \in U$ s.t. $\xi_1 = \frac{\partial}{\partial x^1}$



Pf We can reduce to case $M = \mathbb{A}^m$ with coords $t^1, \ldots, t^m$ and assume $\xi_1(0) = \frac{\partial}{\partial t^1}$. Idea: unique integral curve of ξ_1 through any point $(0, t^2, \ldots, t^m)$. For q on this curve take $(x^1, \ldots, x^m) = (c, t^2, \ldots, t^m)$ where c = time to reach q along the flow.

ie: let ϕ_t be flow generated by ξ_1, then set $\chi(x^1, \ldots, x^m) = \phi_{x^1}(0, x^2, \ldots, x^m)$

Then $d\chi(\partial/\partial x^1) = \xi_1$ (def. of "flow") and $d\chi_0\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial t^i}$ for $i \neq 1$. Thus $d\chi_0 = \mathbb{1}$, ie χ is local diffeo around 0, so χ^{-1} gives a chart around 0, with the desired properties. ∎

But how about two vector fields ξ_1, ξ_2?

An obstruction to realizing $\xi_1 = \partial/\partial x^1$ and $\xi_2 = \partial/\partial x^2$:

Def/Prop For $\xi_1, \xi_2 \in \mathfrak{X}(M)$, $[\xi_1, \xi_2] \in \mathfrak{X}(M)$ is determined by $[\xi_1, \xi_2] f = \xi_1 \xi_2 f - \xi_2 \xi_1 f$.

Pf First on $M = \mathbb{A}^m$:

$\xi_1 = g^i \frac{\partial}{\partial x^i} \quad \xi_2 = h^j \frac{\partial}{\partial x^j}$

$$\begin{aligned} \xi_1 \xi_2 f - \xi_2 \xi_1 f &= (g^i \partial_i)(h^j \partial_j) f - (h^j \partial_j)(g^i \partial_i) f \\ &= g^i h^j \partial_i \partial_j f + \partial_i h^j g^i \partial_j f - h^j g^i \partial_j \partial_i f - h^j \partial_j g^i \partial_i f \\ &= \left[g^j \partial_j h^i - h^j \partial_j g^i \right] \partial_i f \end{aligned}$$

so iff we set $[\xi_1, \xi_2] = \left[g^j \partial_j h^i - h^j \partial_j g^i \right] \frac{\partial}{\partial x^i}$ then we get $(\xi_1 \xi_2 - \xi_2 \xi_1) f = [\xi_1, \xi_2] f$

This formula is enough to determine what happens on general M. But let's spell it out carefully.

Then for general M, with $\xi_1, \xi_2 \in \mathfrak{X}(M)$, pick a chart (x, U) and define $[\xi_1, \xi_2] \in \mathfrak{X}(U)$ by requiring

$$(dx([\xi_1, \xi_2])) f = (dx(\xi_1))(dx(\xi_2)) f - (dx(\xi_2))(dx(\xi_1)) f \qquad f \in C^\infty(x(U))$$

To check independence of chart: take (y, V) another chart. Let $\phi = x \circ y^{-1}$. $\quad x(U) \xleftarrow{\phi} y(U)$

Then $\underset{\overset{\cap}{\mathfrak{X}(y(V))}}{dy([\xi_1,\xi_2])}\, \underset{\overset{\cap}{C^\infty(y(V))}}{(f\circ\phi)} = [d\phi^{-1}(dx([\xi_1,\xi_2]))]\,\phi^* f = \phi^*(dx([\xi_1,\xi_2])f)$

$$\left[\begin{array}{c} d\phi^{-1}(\xi)\,\phi^* f = \phi^*(\xi f) \\ M \xrightarrow{\phi} N \\ \qquad f,\xi \end{array}\right]$$

and similarly

$$dy(\xi_1)\,dy(\xi_2)f - dy(\xi_2)\,dy(\xi_1)f = \phi^*\left[(dx(\xi_1))(dx(\xi_2))f - (dx(\xi_2))(dx(\xi_1))f\right]$$

Thus, get $(dy([\xi_1,\xi_2]))f = (dy(\xi_1))(d(\xi_2))f - (dy(\xi_2))(dy(\xi_1))f$

So $[\xi_1,\xi_2]$ is indeed <u>well defined</u> by this equation ∎

Lemma 1) $[\xi_1,\xi_2] = -[\xi_2,\xi_1]$ 3) in local coords $\left[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right] = 0.$

2) $[\xi_1, f\xi_2] = (\xi_1 f)\,\xi_2 + f[\xi_1,\xi_2]$

<u>Pf</u> 1) easy.

2) $[\xi_1, f\xi_2]g = \xi_1[(f\xi_2)g] - f\xi_2[\xi_1 g] = \xi_1 f\cdot\xi_2 g + f\cdot\xi_1\xi_2 g - f\xi_2\xi_1 g = \xi_1 f\,\xi_2 g + f[\xi_1,\xi_2]g$ ∎

3) $[\partial_i,\partial_j]f = \partial_i\partial_j f - \partial_j\partial_i f = 0.$

<u>Rk</u> This means that if $[\xi_1,\xi_2]\neq 0$ there <u>cannot be</u> a coord sys with $\xi_1 = \partial_1$, $\xi_2 = \partial_2$.

<u>Ex</u> $[y\partial_x, \partial_y + \partial_x] = -\partial_x$

<u>Prop</u> For $\omega\in\Omega^1(M)$, $d\omega\in\Omega^2(M)$ is given by the formula

$d\omega : \mathfrak{X}(M)\times\mathfrak{X}(M)\to C^\infty(M)$

$$d\omega(\xi_1,\xi_2) = \xi_1[\omega(\xi_2)] - \xi_2[\omega(\xi_1)] - \omega([\xi_1,\xi_2]) \qquad \xi_1,\xi_2\in\mathfrak{X}(M)$$

<u>Pf</u> First check this formula actually defines <u>some</u> 2-form. This means checking it's

1) <u>antisymmetric</u> in $\xi_1\leftrightarrow\xi_2$ ✓

2) <u>linear</u> over $C^\infty(M)$:
$$\begin{aligned} & f\xi_1[\omega(\xi_2)] - \xi_2[\omega(f\xi_1)] - \omega([f\xi_1,\xi_2]) \\ &= f\xi_1[\omega(\xi_2)] - \xi_2[f\omega(\xi_1)] - \omega(f[\xi_1,\xi_2] - \xi_2 f\cdot\xi_1) \\ &= f\xi_1[\omega(\xi_2)] - \cancel{\xi_2 f\cdot\omega(\xi_1)} - f\xi_2(\omega(\xi_1)) - f\omega([\xi_1,\xi_2]) + \cancel{\xi_2 f\cdot\omega(\xi_1)} \\ &= f\cdot\left(\xi_1[\omega(\xi_2)] - \xi_2[\omega(\xi_1)] - \omega([\xi_1,\xi_2])\right) \quad ✓ \end{aligned}$$

Then, to check it's actually $d\omega$, enough to check on a <u>basis</u> of sections. So, choose a <u>chart</u> (U,x) and the basis $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1,\dots,m}$. Then if $\omega = f_k dx^k$, $d\omega = \frac{\partial f_k}{\partial x^\ell}dx^\ell\wedge dx^k$, $\omega(\partial_k) = f_k$,

$$d\omega(\partial_i,\partial_j) = \frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j} = \partial_i\omega(\partial_j) - \partial_j\omega(\partial_i) - \omega(\cancel{[\partial_i,\partial_j]}) \qquad ∎$$

<u>Thm</u> (Frobenius)

$\xi_1,\dots,\xi_\ell\in\mathfrak{X}(M)$: if $[\xi_i,\xi_j] = 0$ $\forall i,j = 1,\dots,\ell$ then $\forall p\in M$, $\exists$ chart (x,U) at p such that on U, $\xi_i = \frac{\partial}{\partial x^i}$ $i = 1,\dots,\ell$.

<u>Pf Sketch</u> Same idea as above for $\ell = 1$. <u>But</u>, we need to know that the flows generated by ξ_i <u>commute</u>! We'll show this below.

Def ① $f: M \to N$, $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$: say X and Y are <u>f-related</u> if $\forall p \in X$, $Y(f(p)) = df_p(X(p))$.

② If $f: M \to N$ diffeo, any $X \in \mathfrak{X}(M)$ is f-related to unique elt $f_*X \in \mathfrak{X}(N)$.

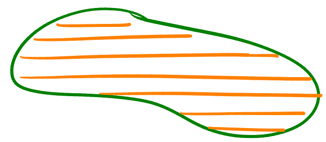
Prop If X, Y are f-related and X', Y' are f-related then $[X, X']$ and $[Y, Y']$ are f-related.

Pf
$$([Y,Y']g)\circ f = (YY'g)\circ f - (Y'Yg)\circ f$$
$$= X(Y'g\circ f) - X'(Yg\circ f) = XX'(g\circ f) - X'X(g\circ f) = [X,X'](g\circ f). \qquad \blacksquare$$

Def ① A <u>distribution</u> D on M is a sub-bundle of TM.

② D is <u>involutive</u> if $\forall X, Y \in \Gamma(D) \subset \mathfrak{X}(TM)$ $[X,Y] \in \Gamma(D)$.

③ D is <u>integrable</u> if M can be covered by charts (x,U) s.t. $x(U) \cap \{x^1 = a^1, \dots, x^l = a^l\}$ is a submanifold $N_{\vec a}$, with $T_pN_{\vec a} = D_p$, $\forall \vec a$.



Cor D is involutive $\Longleftrightarrow$ D is integrable.

Pf ($\Longleftarrow$) is easy.
($\Longrightarrow$) enough to do it locally ie $M \subset \mathbb{A}^m$ open. Can assume $D_0 = \langle \partial_1, \dots, \partial_l \rangle$. Then consider the <u>projection</u> map $\pi: M \to \mathbb{A}^l$; on some nbhd U of 0, $d\pi_p: D_p \xrightarrow{\sim} \mathbb{R}^l$.
Then $\xi_i = d\pi_p^{-1}(\partial/\partial t^i)$ are vector fields on M, and $[\xi_i, \xi_j] \in \Gamma(D|_U)$ is π-related to $\left[\frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j}\right] = 0$, thus $[\xi_i, \xi_j] = 0$. Then use the Frobenius thm. $\blacksquare$

To understand why flows commute, and complete pf of Frobenius thm:

Def $X \in \mathfrak{X}(M)$: ① $\xi \in \mathfrak{X}(M)$: $\mathcal{L}_X\xi \in \mathfrak{X}(M)$ (<u>Lie derivative</u>) given by
$$\mathcal{L}_X\xi(p) = \lim_{t\to 0} \frac{1}{t}\Big[\xi(p) - \overbrace{d(\phi_t)_{\phi_{-t}(p)}\big(\xi(\phi_{-t}(p))\big)}^{[\phi_{t*}\xi](p)}\Big]$$

② $\omega \in \Omega(M)$: $\mathcal{L}_X\omega \in \Omega(M)$ " " $\mathcal{L}_X\omega(p) = \lim_{t\to 0} \frac{1}{t}\left[(\phi_t^*\omega)(p) - \omega(p)\right]$

Ex $M = \mathbb{A}^2$, $X = \partial_x$, $Y = \partial_y$: $\mathcal{L}_X(x\,dy) = \lim_{t\to 0} \frac{1}{t}[(x+t)dy - x\,dy] = dy$, $\mathcal{L}_Y(x\,dy) = \lim_{t\to 0}\frac{1}{t}[x\,d(y+t) - x\,dy] = 0$

Prop $X, \xi \in \mathfrak{X}(M)$: $\mathcal{L}_X\xi = [X, \xi]$

Pf local coordinates: $X = f^i\partial_i$, $\xi = g^j\partial_j$ $\quad p = 0$
$$\mathcal{L}_X\xi(0) = \lim_{t\to 0}\frac{1}{t}\left[d(\phi_{-t})_0(\xi(0)) - \xi(\phi_{-t}(0))\right] \qquad \phi_{-t}(x)^i = x^i - tf^i(x) + \cdots \qquad \lim_{t\to 0}\frac{(\dots)}{t} = 0$$
$$= \lim_{t\to 0}\frac{1}{t}\left[g^j(0)\partial_j - t\frac{\partial f^j}{\partial x^i}g^i\partial_j + \cdots - g^j(-tf^i(0) + \cdots)\partial_j\right]$$
$$= \left[-g^i\partial_i f^j + f^i\partial_i g^j\right](0)\,\partial_j \qquad \blacksquare$$

Lemma (remedial): $f: M \to N$, $\phi: I \to M$: $\underset{\substack{\uparrow \\ TN}}{\frac{d}{dt}(f(\phi(t)))} = df_{\phi(t)}\Big(\underset{\substack{\uparrow \\ TM}}{\frac{d}{dt}\phi(t)}\Big)$

Pf Chain rule.

Prop $f: M \to N$, $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ f-related, ϕ_t flow of X, $\tilde{\phi}_t$ flow of Y: $\tilde{\phi}_t \circ f = f \circ \phi_t$

Pf They agree at $t=0$, and $\frac{d}{dt}\tilde{\phi}_t(f(p)) = Y(\tilde{\phi}_t(f(p)))$, $\frac{d}{dt}\left(f(\phi_t(p))\right) = df_{\phi_t(p)}(X(\phi_t(p))) = Y(f(\phi_t(p)))$
ie both sides obey the same 1st-order ODE

$$\begin{tikzcd} M \arrow[loop left, "\phi_t"] \arrow[r, bend left, "f"] & N \arrow[l, bend left, "f^{-1}"] \arrow[loop right, "\tilde{\phi}_t"] \end{tikzcd}$$

∎

Prop $X, Y \in \mathfrak{X}(M)$, $[X,Y] = 0$, ϕ_t flow of X, ψ_t flow of Y: $\phi_t \circ \psi_s = \psi_s \circ \phi_t$.

Pf Set $f = \psi_s$ in the above prop^n. So want to show X is ψ_s-related to itself, i.e. $X = \psi_{s*}X$.

Clearly true for $s=0$, so consider

$$\frac{d}{ds}\psi_{s*}X(p) = \lim_{h\to 0}\frac{\psi_{s+h*}X(p) - \psi_{s*}X(p)}{h}$$

$$= \lim_{h\to 0}\psi_{s*}\left(\frac{\psi_{h*}X(p) - X(p)}{h}\right)$$

$$= \lim_{h\to 0}(d\psi_s)_{\psi_{-s}(p)}\left[\frac{(d\psi_h)_{\psi_{-s-h}(p)}X(\psi_{-s-h}(p)) - X(\psi_{-s}(p))}{h}\right]$$

$$= (d\psi_s)_{\psi_{-s}(p)}\left[\mathcal{L}_Y X(\psi_{-s}(p))\right] = 0$$

as desired. ∎

Finally this finishes pf of Frobenius thm.