Integrate scalar in $d=4$

Scaler field with $S = \frac{1}{2} \int \nabla \Phi^2 + \frac{\mu^2}{2} \Phi^2 + \frac{\Lambda}{4!} \Phi^4$ \( \mathcal{C} = \{ \Phi: \mathbb{R} \rightarrow \mathbb{R} \} \)

$$\langle \Phi(x_1) \Phi(x_2) \rangle = \sum_{x_1} + \sum_{x_2} + \ldots \quad = D(x_1, x_2) + \frac{\Lambda}{2} \int dx'D(x_1, x') D(x_2, x') D(x', x') + \ldots$$

Here $D(\cdot, \cdot)$ is the Green's function for the Laplacian: $(\Delta_{x_1} + \mu^2) D(x_1, x_2) = \delta(x_1, x_2)$

If $X = \mathbb{R}^4$ flat and $m=0$, $D(x, y) = \frac{1}{||x-y||^2}$ (cf. $X = \mathbb{R}^4$, $D(x, y) = \frac{1}{2} |x-y|$)

so $D(x; x')$ is infinite! (even if $m \neq 0$)

How to interpret this $\infty$?

First observe: it "comes from the high energy part of the theory"! e.g. on flat $\mathbb{R}^4$,

$$D(x, y) = \int_{(\mathbb{R}^4)^*} dp \frac{e^{ip \cdot (x-y)}}{||p||^2 + m^2} \quad \text{so} \quad D(x, x) = \int_{||p|| > \Lambda} dp \frac{1}{||p||^2 + m^2}$$

A way to remove the problem: view our action $S$ not as fundamental but as an effective action.

So, path integrals run only over $\mathcal{C}_\Lambda = \{ \Phi: \mathbb{R}^4 \rightarrow \mathbb{R} \}$

In the perturbative analysis of path integrals over $\mathcal{C}_\Lambda$, $D(x, y)$ gets replaced by a cutoff version:

$$D_\Lambda(x, y) = \int_{||p|| < \Lambda} dp \frac{1}{||p||^2 + m^2} e^{ip \cdot (x-y)}$$
This renders answers finite (at least if $m \neq 0$).

But they depend on $\Lambda$. If $\Lambda \gg E$ the energy scale we want to study, this will cause problems: for example,

$$
\langle \phi(x_1) \phi(x_2) \rangle = D(x_1, x_2) + \left[ (\#) \cdot \Lambda \cdot \Lambda \| x \|^2 + \cdots \right] + \cdots
$$

So even if the coupling $\lambda$ is small, this perturbation expansion may not be well behaved.

That might be OK. After all, $\Lambda$ isn't directly observable anyway.

We should try instead to formulate things in terms of even more effective action $S_{\text{eff}}(E)$, where we've integrated out modes between $\Lambda$ and $E$. What kind of terms will occur?

General expectation: $S_{\text{eff}}(E)$ is non-local. Expanding yields an infinite series of interactions. That seems terrible: how could you ever use such a theory for any practical purpose?

To investigate more closely: convenient to integrate out in way that lowers $\Lambda$ "continuously".

This can be done precisely ("exact RGE"):

writing the interaction part $S_{\text{int}}$, cutoff function $\rho$,

$$
\Lambda \frac{\partial S_{\text{eff}}}{\partial \Lambda} = - \int d^4 p \left( \frac{\pi}{2} \right)^4 \frac{1}{(p^2 + m^2)} \left[ \frac{\partial}{\partial \Lambda} \rho \left( \frac{p^2 + m^2}{\Lambda^2} \right) \left[ \frac{\partial S_{\text{int}}}{\partial \phi(p)} \frac{\partial S_{\text{int}}}{\partial \phi(p)} + \frac{\partial^2 S_{\text{int}}}{\partial \phi(p) \partial \phi(p)} \right] \right]
$$

This defines a flow in the $\infty$-dim space of possible Lagrangians.

Key idea: as $\Lambda \to 0$ this flow is driven to a 3-dimensional subspace!

i.e. if we start from a very high scale $\Lambda$ and flow down to $\Lambda_0$,

there is only a 3-dimensional space of possible effective theories that we can get, up to corrections which are suppressed by powers of $\Lambda_0 / \Lambda$. 

Polchinski: "Renormalization and effective Lagrangians"

(Also Costello)
This is why QFT has some power: after measuring finitely many parameters, everything else is determined (and in a computationally effective way...)

To understand why this should be, we should think a little about scaling.
There's an action $\rho_\varepsilon$ of the group $\mathbb{R}^\times$ on $\Omega$ by $\phi(x) \mapsto \frac{1}{\varepsilon} \phi(\varepsilon x)$
This action was chosen so that it leaves the kinetic term $\frac{1}{2} \int \|d\phi\|^2$ invariant.
But it transforms the other terms,
\[
\rho_\varepsilon^* (\int \phi^n) = \varepsilon^{-n} \int \phi^n \\
\rho_\varepsilon^* (\int \|d\phi\|^m \phi^n) = \varepsilon^{-2m} \int \|d\phi\|^m \phi^n
\]
Define the scalar dimension of the coupling:
\[
\dim \phi^n = n \\
\dim \|d\phi\|^m \phi^n = 2m+n
\]
\[
\left[ \frac{1}{\varepsilon^2} \langle \phi(0) \phi(\varepsilon x) \rangle \text{ computed with action } S \right] = \left[ \langle \phi(0) \phi(x) \rangle \text{ with } \rho_\varepsilon^* (S) \right]
\]
\[\implies \text{ at least naively (if we're not thinking about role of cutoffs), we'd say that for large } \varepsilon \text{ (large distance) the effects of the terms with } \dim > 4 \text{ become small. Call these terms "irrelevant."}
\]
More generally:
\[
\begin{align*}
\dim > 4 & \quad \text{irrelevant} \\
\dim = 4 & \quad \text{marginal} \\
\dim < 4 & \quad \text{relevant}
\end{align*}
\]
There are 3 marginal or relevant terms possible which are invariant under $\phi \mapsto -\phi$ and invariant under the Poincare group: $\phi^2, \phi^4, \|d\phi\|^2$.

This is really just dimensional analysis so far.
Another way to express the same idea: the terms in $S$ will always contribute to correlators with some powers of $E$ to make them dimensionless. So we should measure them...
that way: \[ S_{\text{int}}(\Lambda) = g_4 \phi^4 + g_6 \Lambda^2 \phi^6 + g_8 \Lambda^4 \phi^8 + \cdots \]

Then, the **classical running** would be \[ \Lambda \frac{dg_4}{d\Lambda} = 0, \quad \Lambda \frac{dg_6}{d\Lambda} = 2g_6, \ldots \]

The **exact running** is \[ \Lambda \frac{dg_4}{d\Lambda} = \bar{\beta}_4 (g_4, g_6, \ldots) \]

\[ \Lambda \frac{dg_6}{d\Lambda} = 2g_6 + \bar{\beta}_6 (g_4, g_6, \ldots) \]

\[ \vdots \]

Studying toy examples we see that the initial value of \( g_6 \) (and all other irrelevant couplings) gets **damped out** exponentially fast as \( \log \Lambda \) decreases.

It's proven that this really happens in this theory (at least in perturbation) [Polchinski].

Even after the flow converges onto the finite-dim. subspace spanned by marginal and relevant couplings, at \( \Lambda \ll \Lambda_0 \), we still have a nontrivial flow of couplings with \( \Lambda \). Coordinating by \( S = \frac{1}{2} \tilde{Z}_2 \| d\Phi \|^2 + \frac{1}{2} m^2 \phi^2 + \frac{g_4}{4!} \phi^4 + \cdots \)

\[ \Lambda \frac{dg_4}{d\Lambda} = \beta_4 (g_4, \frac{m^2}{\Lambda^2}, \tilde{Z}_2) \]

\( \beta_k \) can be computed in perturbation theory, e.g. in the limit \( m \to 0 \) and \( \Lambda_0 \to \infty \),

\[ \beta_4 = \frac{3g_4^2}{16\pi^2} + O(g_4^3) \]

Morally, this comes from the relation: (using = for lines integrated only over scales between \( \Lambda \) and \( \Lambda' \))

\[ \begin{array}{cccc}
\times & = & \times & + \times + \times + \cdots \\
\times & + & \times & + \times & + \times & + \cdots
\end{array} \]

\[ g_4(\Lambda') = g_4(\Lambda) + 3 \cdot \frac{1}{2} \left( \begin{array}{c}
\int_{\Lambda' < |q| < \Lambda} \frac{dp}{2\pi^2} q^4
\end{array} \right) = g_4(\Lambda) + \frac{3g_4(\Lambda)^2}{16\pi^2} (\log \Lambda - \log \Lambda') + \cdots \]

For a systematic treatment see [Hughes-Lin]
This says that the effective theory (always at scales $\Lambda \ll \Lambda_0$) becomes more strongly coupled when we increase $\Lambda$.

$\Rightarrow$ perturbative theory becomes less and less effective as we go to higher energies.

For any finite $\Lambda_0$, the theory exists, but it’s hard to understand at energies where $g_\Lambda \gtrsim 1$.

We don’t really understand whether it is possible to remove the cutoff completely,

i.e. to take $\Lambda_0 \rightarrow \infty$ while holding $g_\Lambda(\Lambda)$ fixed.

Usually summarized by saying this theory exists “only as an effective theory”.

This eff. theory is IR free: effect of interactions disappears at low enough energies.