Twisting

We construct our SUSY gauge theory on general Riemannian \( X \).
\( \text{ISO}(4) \) replaced by \( \text{Isom}(X) \). \((\text{ISp}(4)\text{ rep} \text{ by } \text{gp of spin isomorphs.})\)

But now ask: what are the analogues of the odd vector fields \( Q_x \)?
For any \( \frac{\zeta}{\xi} \in T^*(S^+_B \otimes S^-_B) \) we can write an odd u.d., but it annihilates the action
\( S \) only if \( \nabla \frac{\zeta}{\xi} = 0 \). What can we do on an arbitrary \( X \), maybe w/ no isomorphs?

Idea of twisting: replace the "R-symmetry vector space" \( R \) by
an \( \text{SU}(2) \) vector bundle over \( X \), with a fixed (non-dynamical) connection.

To get this bundle: fix a homomorphism \( i : \text{Sp}(4) \rightarrow \text{SU}(2) \)
Using \( i \), the rep \( R \) of \( \text{SU}(2) \) induces a rep \( R^t \) of \( \text{Sp}(4) \).

Now, we write exactly the same action we wrote before, just replacing \( R \rightarrow (R^t)_B \)
with \( B \) the \( \text{Sp}(4) \)-bundle over \( X \) given by the spin structure, and using covariant
derivatives where needed.

Our new field space:
\[
\mathcal{C}_X = \begin{cases} 
\left( P, V \right) : & \text{principal } G \text{-bundle w/can over } X \\
\nu^x \in T^* \left( S^+_B \otimes R^t \right)_B \otimes \mathbb{C} \otimes \mathbb{C} \\
\phi \in T^* \left( \mathbb{C} \otimes \mathbb{C} \right) \\
D \in T^* \left( \mathbb{C} \otimes \text{Sym}^2 \left( R^t \right)_B \right)
\end{cases}
\]

Let's choose:
\( i : \text{SU}(2)_+ \times \text{SU}(2)_- \rightarrow \text{SU}(2) \)
\( (g_+, g_-) \rightarrow g_+ \)

Then \( R^t = S^+ \).
So really the effect of twisting is "replace \( R \) with \( S^+ \) everywhere."

In \( p^t \), now look at our odd vector fields. They were generated by \( \frac{\zeta}{\xi} \in T^*(S^+_B \otimes S^-_B) \otimes R \).
Twist replace that with \( \frac{\zeta}{\xi} \in T^* \left( S^+_B \otimes S_B^+ \otimes S^-_B \otimes S^+_B \right)_B \).
\[
\Gamma^\prime((\mathbb{C} \oplus \text{Sym}^2(S^+)) \oplus \text{fund})_g
\]

\[
= \Gamma^\prime((1,1) \oplus (3,1) \oplus (2,2))_g \quad \text{(in physicists' notation)}
\]

Note, one trivial summand! This trivial bundle does have a c.c. section, no matter what X is.

\[\Rightarrow\] the twisted version of the theory has a single odd vector field \(Q\) on \(\Sigma\), \(QS = 0\).

Also, one summand \(\sim TX\), giving v.f. \(Q_v\) for \(v \in \Gamma^\prime(TX)\). (But \(Q_S \neq 0\) generally)

Let's write the action now in twisted notation:

\[\mathcal{L} = \begin{cases}
(P, \nabla): \text{principal } G\text{-bundle w/conn over } X \\
\chi \in \Gamma(\mathcal{G}_\text{CP} \otimes \text{fund}^\ast)
\end{cases}
\]

\[= \begin{cases}
\chi \in \Gamma(\mathcal{G}_\text{CP} \otimes \text{Sym}^2(S^+))
\end{cases}
\]

\[\phi \in \Gamma(\mathcal{G}_\text{CP})
\]

\[D \in \Gamma(\mathcal{G}_\text{CP} \otimes \text{Sym}^2(S^+))
\]

\[
S = \frac{1}{g^2} \int_X \text{Tr} \left( \frac{1}{2} \nabla^2 \phi - i \langle \chi, \nabla \phi \rangle - i \chi^\ast \nabla \phi - \frac{1}{2} ||F||^2 \right.
\]

\[+ \frac{i}{g} ||D\phi||^2 - \frac{i}{2} [\phi, \phi]^2 - \frac{i}{2} \langle \chi, [\phi, \phi] \rangle
\]

\[+ i \sqrt{2} \eta [\phi, \phi] - \frac{i}{2} \langle \chi, [\phi, \phi] \rangle \bigg) \]

The odd vector field act by:

\[
\delta \phi = 0 \quad \delta \overline{\phi} = \varepsilon 2 \sqrt{2} i \chi
\]

\[
\delta A = \varepsilon \chi
\]

\[
\delta X = i \varepsilon (F^+ - D)
\]

\[
\delta \eta = \varepsilon [\phi, \overline{\phi}]
\]

\[
\delta D = \varepsilon (2\nabla \chi + 2 \sqrt{2} \varepsilon [\phi, \chi])
\]

Fixed point: \(F^+ = D, \nabla \phi = 0, [\phi, \overline{\phi}] = 0\).

But critical points have \(D = 0\) so we expect behavior to \(F^+ = 0, \nabla \phi = 0\) for any observables that are annihilated by \(\delta\).
What does the eq. $\nabla \phi = 0$, $[\phi, \bar{\phi}] = 0$ mean? If it has a solution then we have an inf\textsuperscript{t} automorphism of $(P, \nabla)$.

Decompose $f$ under the action of $\phi$: $[\phi, \bar{\phi}] = 0 \Rightarrow \text{each fiber splits into } \Theta$ of 2 lines with opposite eigenvalues $\lambda, -\lambda$. $\nabla \phi = 0 \Rightarrow \lambda$

is constant $\Rightarrow$ we get a global decomposition $(P, \nabla) = (\mathbb{C}, \nabla') \oplus (\mathbb{C}', \nabla'')$ of two $U(1)$ bundles w/connections. ("reducible connection.")

If also $F^+ = 0$ then $(P, \nabla)$ is a reducible instanton.

Source of headaches in Donaldson theory since they have larger-than-usual stabilizers in $G \Rightarrow$ give singularities on the moduli space of instantons!

We'd like to avoid these problems. NB: if we have a reducible instanton then $(\mathbb{C}, \nabla')$ is a $U(1)$ instanton. $F_{\nabla'} \in \Omega^{2, -1}(X)$

$\Rightarrow \frac{1}{2\pi} [F_{\nabla'}] \in H^{2, -1}(X) \cap H^2(X, \mathbb{Z})$

The existence of any nonzero elt in $H^{2, -1}(X) \cap H^2(X, \mathbb{Z})$ is a constraint on $(X, g)$:

as long as $b^+_2(X) \geq 1$, a "generic" lattice would miss $H^{2, -1}(X)$.

So, let's suppose $b^+_2(X) \geq 1$ and $g$ is "generic." Then we would expect localization to the space of sol's of $F^+ = 0$, modulo gauge.

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What are the good observables? Need $O$ with $Q(O) = 0$.

Simplest: $O^{(2)}(x) = Tr \phi^2(x)$. Indeed has $Q(O^{(2)}(x)) = 0$.

To get others: use the odd v.f. $G_v$ assoc. to vector fields $v$ on $X$. 

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\begin{itemize}
    \item What does this eq. $\nabla \phi = 0$, $[\phi, \bar{\phi}] = 0$ mean?
    \item If it has a solution then we have an inf\textsuperscript{t} automorphism of $(P, \nabla)$.
    \item Decompose $f$ under the action of $\phi$: $[\phi, \bar{\phi}] = 0 \Rightarrow \text{each fiber splits into } \Theta$ of 2 lines with opposite eigenvalues $\lambda, -\lambda$. $\nabla \phi = 0 \Rightarrow \lambda$
    \item is constant $\Rightarrow$ we get a global decomposition $(P, \nabla) = (\mathbb{C}, \nabla') \oplus (\mathbb{C}', \nabla'')$ of two $U(1)$ bundles w/connections. ("reducible connection.")
    \item If also $F^+ = 0$ then $(P, \nabla)$ is a reducible instanton.
    \item Source of headaches in Donaldson theory since they have larger-than-usual stabilizers in $G \Rightarrow$ give singularities on the moduli space of instantons!
    \item We'd like to avoid these problems. NB: if we have a reducible instanton then $(\mathbb{C}, \nabla')$ is a $U(1)$ instanton. $F_{\nabla'} \in \Omega^{2, -1}(X)$
    \item $\Rightarrow \frac{1}{2\pi} [F_{\nabla'}] \in H^{2, -1}(X) \cap H^2(X, \mathbb{Z})$
    \item The existence of any nonzero elt in $H^{2, -1}(X) \cap H^2(X, \mathbb{Z})$ is a constraint on $(X, g)$:
    \item as long as $b^+_2(X) \geq 1$, a "generic" lattice would miss $H^{2, -1}(X)$.
    \item So, let's suppose $b^+_2(X) \geq 1$ and $g$ is "generic." Then we would expect localization to the space of sol's of $F^+ = 0$, modulo gauge.
    \item What are the good observables? Need $O$ with $Q(O) = 0$.
    \item Simplest: $O^{(2)}(x) = Tr \phi^2(x)$. Indeed has $Q(O^{(2)}(x)) = 0$.
    \item To get others: use the odd v.f. $G_v$ assoc. to vector fields $v$ on $X$. 
\end{itemize}
\[ \delta \phi = \frac{1}{2i} \langle \nu, \phi \rangle \]
\[ \delta A_\mu = \frac{i}{2} (g_{\mu \nu} \chi - i X_{\mu \nu}) \nu \]
\[ \delta \gamma = -i \sqrt{2} \nabla \phi \]
\[ \delta D_\mu = -(F_{\mu \nu} + D_{\mu \nu}) \nu \]
\[ \delta \phi = 0 \]
\[ \delta X = -\frac{3i\sqrt{2}}{8} * \nabla \bar{\phi} \]
\[ \delta D = -\frac{3i}{4} * \nabla \gamma + \frac{3i}{2} \nabla \phi \]

It obeys \[ [Q, G_v] = P_v \]

So: if we define "descent" by
\[ \sigma^{(k+1)}(x) = \sum_m \int \frac{G_m}{2} \sigma^{(k)}(x) \wedge dx^m \]

then \[ Q(\sigma^{(1)}(x)) = \frac{\partial}{\partial x^\mu} \sigma^{(0)}(x) \]

simply,
\[ Q(\sigma^{(k)}(x)) = k \sigma^{(k+1)}(x) \]

Hence, if \( Y \) is a k-cycle on \( X \), and we define \[ \sigma^{(k)}(Y) = \int_Y \sigma^{(k)}(x) \]

then we have \[ Q(\sigma^{(k)}(Y)) = 0 \]

We'll only use \( k = 0, 1, 2 \).

So, we expect localization for observables of the form
\[ \langle \sigma^{(0)}(x_1) \sigma^{(0)}(x_2) \ldots \sigma^{(1)}(x_1) \sigma^{(1)}(x_2) \ldots \sigma^{(2)}(x_1) \sigma^{(2)}(x_2) \ldots \rangle \]

**Deformation invariance**

As we had in previous examples, we expect \( \langle Q \sigma \rangle = 0 \) for any \( \sigma \).

In particular: \[ \langle \sigma^{(0)}(x_1) \rangle - \langle \sigma^{(0)}(x_2) \rangle \]
\[ = \int_{x_1}^{x_2} \langle Q(\sigma^{(1)}(x)) \rangle = 0 \]

So, \( \langle \sigma^{(0)}(x) \rangle \) should be independent of \( x \): just write it \( \langle \sigma^{(0)} \rangle \)
In a similar way, \(<\sigma^{(k)}(x)\>) should depend only on the homology class \([y] \in H_k(X, \mathbb{Z})\).

Also, the whole action has the nice property

\[
S = \frac{1}{g^2} \left[ Q(V) - \frac{1}{2} \int_x \text{Tr} \, F \wedge F \right] + \frac{g}{4\pi^2} \int_x \text{Tr} \, F \wedge F
\]

\[
= \frac{1}{g^2} Q(V) + \frac{i \pi}{4\pi} \int_x \text{Tr} \, F \wedge F \quad \left[ \tau = \frac{\vartheta}{\pi} + \frac{i}{2g^2} \right]
\]

where \(V = \int \text{Tr} (\frac{i}{4} \langle x, F + D \rangle - \frac{1}{2} \eta [\phi, \bar{\phi}] + \frac{1}{2g^2} \phi \nabla \bar{\phi})\)

In particular, all the metric dependence is in the term \(Q(V)\)!

So, at least formally we expect that all \(Q\)-invariant correl. func. are indep. of metric on \(X\), i.e., this is a "topological quantum field theory," in physicists' sense.

Very different from the usual kind of field theory.