Coupling to fermions

Recall the spin representation of $\text{Cliff}(\mathbb{R}^n)$, $S \simeq S^+ \oplus S^-$. $\text{Spin}(\mathbb{R}^n) \simeq SU(2)_+ \times SU(2)_-$ and in fact $S^+$ is the fundamental representation of $SU(2)_+$ and $S^-$ is the dual representation of $SU(2)_-$.

Now if $X$ is a spin $4$-manifold (with principal $\text{Spin}(4)$ bundle $B$) then we get associated spin bundles $S^\pm_B$. Dirac operator $\hat{\gamma} : \Gamma(S^+_B) \to \Gamma(S^-_B)$.

Now we can define a new theory by $
\mathcal{C} = \begin{cases} 
\frac{1}{2} \int_X \langle \gamma^+, \gamma^+ \rangle 
\end{cases}$

$\langle , \rangle =$ Hermitian inner product on $S^+$

That's free (quadratic).

Can also couple to gauge fields: $\mathcal{C} = \begin{cases} 
(\mathcal{D}) 
\frac{1}{2} \int_X \langle \gamma^+, \gamma^+ \rangle 
\end{cases}$

$\mathcal{D} =$ covariant Dirac op:

$\sum_i \rho(e_i) D_{e_i} \quad (e_i \text{ orthonormal basis})$

[Our rules for path integration over these fermions will be by analytic continuation from signature $(3,1)$.

It's a general principle of QFT that fermions usually "have $\frac{1}{2}$-integer spin" i.e. they are associated with reps of $\text{so}(\mathbb{R}^n)$ that don't integrate to $\text{SO}(\mathbb{R}^n)$ but only to $\text{Spin}(\mathbb{R}^n)$. "Spin-statistics theorem" (but has loopholes...)

Roughly a consequence of unitarity.