Supersymmetric gauge theory

We are now going to write a theory which extends the Yan-Mills action. Fix an auxiliary \( R \)-symmetry vector space \( R \cong \mathbb{C}^2 \).

\[ (P, \mathcal{O}) : \text{principal } G \text{-bundle w/conn over } X \]

\[ \lambda^\pm \in T'(S^\pm \otimes \mathcal{O}_{C, P} \otimes R) \]

\[ \phi \in T'(\mathcal{O}_{C, P}) \]

\[ D \in T'(\mathcal{O}_{C, P} \otimes \text{Sym}^2(R)) \]

Wrt a basis of \( R \), expand \( \lambda^\pm \) as pair \( \lambda_1^\pm, \lambda_2^\pm \in T'(S^\pm \otimes \mathcal{O}_{C, P}) \), also expand \( D \) as triplet \( D_{11}, D_{12}, D_{22} \); let \( \delta \) denote herm. metric in \( R \), \( \varepsilon \in \wedge^2(R) \) of unit norm.

\[
S = \frac{1}{g^2} \int_X \left( -\frac{1}{4} F_{\lambda^+} \ast F + \nabla^\lambda \phi \nabla^\lambda \phi - i \varepsilon_{\lambda^+} \left< \lambda^+, \nabla \lambda^+ \right> \right.
\]

\[
+ \frac{1}{4} \delta_{\lambda^+} D_{\lambda^+} D_{\lambda^+} - \frac{1}{2} \left[ \phi, \overline{\phi} \right]^2 - i \sqrt{2} \varepsilon_{\lambda^+} \left< \lambda^+, [\overline{\lambda}, \lambda] \right> + i \sqrt{2} \varepsilon_{\lambda^+} \left< \lambda^+, [\phi, \overline{\lambda}] \right> \right)
\]

\[
+ \frac{i}{4\sqrt{2}} \int_X \text{Tr}(F \ast F)
\]

\[
\text{usual minimally coupled kinetic terms}
\]

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(If we analytically continued to Minkowski signature and put \( \lambda^+ = \overline{\lambda} \), then this would be naturally real.) Note \( D \) enters quadratically and can be integrated out for free ("auxiliary field") but it's convenient to keep it around, as we'll see.

This action has a lot of symmetries:

- gauge symmetry \( G \)
- Lorentz symmetry \( \text{Spin}(4) \)
- translation vector fields \( P \) for \( v \in (T^* \mathbb{R}^4) \)
- "\( R \)-symmetry" \( SU(2) \) acting on \( \lambda^\pm \) and \( D \) (via its action on \( R \))
- "\( R \)-symmetry" \( U(1) \) acting by \( \lambda^\pm \mapsto e^{2i\theta} \lambda^\pm, \phi \mapsto e^{2i\theta} \phi \)
- odd symmetries: vector fields \( Q_3 \) for \( 3 \in (S^+ \otimes S^-) \otimes R \)

\[ \Rightarrow \text{for an infinitesimal param. } \xi \text{ we get var.} \]

\[ \frac{\delta S}{\delta \phi} = \text{e.g. } \frac{1}{g} \nabla^\lambda \phi \nabla^\lambda \phi - \frac{1}{4} \varepsilon_{\lambda} \left< \lambda, \nabla \lambda \right> + \text{other terms} \]
\[ \delta \phi = \sqrt{2} \varepsilon_{\nu} \xi_{\nu}^+ \xi_{\nu}^- \]
\[ \delta A = \delta \nabla^\nu (\xi_{\nu}^+ \sigma_\mu \xi_{\nu}^-) - i \xi_{\nu}^+ \phi \overline{\phi} + i \left[ \sigma_\mu, \xi_{\nu}^- \xi_{\nu}^+ \right] F^\mu \]
\[ \delta \xi_{\nu}^\pm = \delta \nabla^\nu \xi_{\nu}^\pm - i \xi_{\nu}^\pm [\phi, \overline{\phi}] + i \left[ \sigma_\mu, \xi_{\nu}^- \xi_{\nu}^+ \right] \xi_{\nu}^\pm F^\mu \]
\[ \pm i \sqrt{2} \varepsilon_{\nu} \sigma^\mu \xi_{\nu}^\pm D^\mu \phi \]
\[ \delta D_{\nu} = i \left[ \xi_{\nu}^-, \delta \xi_{\nu}^+ \right] + i \sqrt{2} \xi_{\nu}^+ [\phi, \overline{\phi}] + i \sqrt{2} \xi_{\nu}^- [\phi, \overline{\phi}] + (\nu \leftrightarrow \omega) \]

These vector fields have \( \{ Q_{\xi_1}, Q_{\xi_2} \} = P_{T(\xi \xi)} \)

where \( T \) is a map of \( Sp(n) \) reps,

\[ T: (S^+ S^{-}) \otimes (S^+ S^{-}) \rightarrow V \quad \left[ V = \text{fundamental rep of } SO(4) \right] \]

\[ [\text{induced from } S^+ S^- \rightarrow V \text{ and } R \otimes R \rightarrow V] \]

Because of these odd symmetries we will expect some nice localization for computing invariant observables. For example:

\[ \delta \xi_{\nu}^\pm = 0 \] would say

\[ \delta \nabla^\nu \xi_{\nu}^\pm - i \xi_{\nu}^\pm [\phi, \overline{\phi}] + i \left[ \sigma_\mu, \xi_{\nu}^- \xi_{\nu}^+ \right] \xi_{\nu}^\pm F^\mu \pm i \sqrt{2} \varepsilon_{\nu} \sigma^\mu \xi_{\nu}^\pm D^\mu \phi = 0 \]

If we set \( D_{\nu} = 0 \) (as we should if we're interested in minimization of \( S \))

and also suppose \([\phi, \overline{\phi}] = \nabla \phi = 0\)

then this says \( \mathcal{F} \xi_{\nu}^\pm = 0 \) \((\mathcal{F} = [\sigma_\mu, \xi_{\nu}^\pm] F^\mu, \mathcal{F} : S^\pm \rightarrow S^\pm)\)

(i.e. \( \mathcal{F}(x) \xi_{\nu}^\pm = 0 \) \( \forall x \in \mathbb{R}^4 \))
Now we can ask: for which $F$ does $F$ annihilate some elt $\xi \in S^+ \oplus S^-$?

Answer: this happens only if $F$ is either self-dual or anti-self-dual!

\[
\begin{bmatrix}
F & \rightarrow & 0 \\
\Lambda^2(\mathbb{R}^4) & \underset{\text{II}}{\rightarrow} & \text{So}(4) \\
\Lambda^2(\mathbb{R}^4)^+ \oplus \Lambda^2(\mathbb{R}^4)^- & \underset{\text{II}}{\rightarrow} & \text{su}(2) \times \text{su}(2)
\end{bmatrix}
\]

So, if we compute an observable that is annihilated by some $Q_{\pm}$ ($Q_{\mp}$) we'd expect localization to moduli space of instantons (anti-instantons) on $\mathbb{R}^4$.

"Nekrasov formula" is of this sort, for a cleverly chosen observable...

But our interest now is in computing on some compact $X$, not on $\mathbb{R}^4$.

For this, we'll need to make a non-obvious modif" of the action...