Exercise 1

Directly verify two assertions from class:
1. If $M = \mathbb{R}^n$ with its usual flat metric, the Laplacian $\Delta$ acting on differential forms simply acts by $\Delta(\sum_I f_I dx_I) = \sum_I \Delta(f_I) dx_I$ where the $\Delta$ on the right is the usual Laplacian acting on functions.
2. If $X = \mathbb{C}^n$ with its usual flat metric, then $\Delta_\theta = \Delta_{\bar{\theta}} = \frac{1}{2} \Delta$.

Exercise 2

Prove the local version of the $\partial \bar{\partial}$-lemma for real $(1,1)$ forms: if $\omega \in A^{1,1}(U) \cap A^{2}(U)$ for $U \subset \mathbb{C}^n$ a polydisc, with $d\omega = 0$, then $\omega = \partial \bar{\partial} \gamma$ for some $\gamma \in A^0(U)$. (The only analytic inputs needed are the $\bar{\partial}$-Poincare lemma and the ordinary Poincare lemma.)

Exercise 3

This exercise fills in one of the steps I omitted in the proof that the Levi-Civita connection on a Kähler manifold preserves the complex structure operator $I$.

Suppose $X$ is a complex manifold with a Hermitian metric $h$. Let $\nabla$ be the Levi-Civita connection. Define $A(X, Y, Z) = h(I(\nabla_X Y) - (\nabla_X I)Y, Z)$.
1. Show that $A(X, Y, Z) = A(Y, X, Z)$. (Hint: use Exercise 3 of set 2, the vanishing of the Nijenhuis tensor.)
2. Show that $A(X, Y, Z) = -A(X, Z, Y)$.
3. Conclude that $A = 0$ and hence that $I(\nabla_X Y) = (\nabla_X I)Y$.

Exercise 4

Suppose $X$ is Kähler and $\alpha$ is a closed $(1,1)$-form which is primitive (i.e. $\Lambda(\alpha) = 0$).
Show that $\Delta \alpha = 0$.

Exercise 5

1. Suppose $V$ is a vector space with compatible inner product, complex structure and fundamental form $(g, I, \omega)$. Suppose $W \subset V$ is a subspace of dimension $2m$. Choose an orientation on $W$; together with $g$ this induces a volume form $\text{vol}_W$. Show that $\text{vol}_W/\omega^m|_W \geq \frac{1}{m!}$, with equality if and only if $W$ is a complex subspace of $V$, i.e. if $IW = W$.
2. Suppose $X$ is a Kähler manifold and $Y$ a compact submanifold of dimension $2m$. Show that $\text{vol}(Y) \geq \int_Y \omega_m^{2m/\text{dim} Y}$, with equality if and only if $Y$ is a complex submanifold of $X$.
3. Suppose $X$ is a Kähler manifold for which $\omega$ is exact ($\omega = d\alpha$ for some $\alpha$). Show that $X$ has no compact complex submanifolds (in particular $X$ is not compact).
Exercise 6

Let $X$ be a Kähler manifold. Let $I : \mathcal{A}(X) \to \mathcal{A}(X)$ be the standard extension of $I : TX \to TX$. Show that $[I, d] = d^c$ and $[I, d^c] = -d$. 