Complex Geometry: Exercise Set 4

Exercise 1
1. Suppose $M$ is an oriented Riemannian manifold of dimension $n$. Verify the assertion from class that $\ast^2 = (-1)^{k(n-k)}$ acting on $\Omega^k(M)$.
2. If $M = X$ is complex, show that $\ast^2 = (-1)^k$ acting on $\Omega^k(X)$.

Exercise 2
Directly verify two assertions from class:
1. If $M = \mathbb{R}^n$ with its usual flat metric, the Laplacian $\Delta$ acting on differential forms simply acts by $\Delta(\sum_i f_i dx_i) = \sum_i \Delta(f_i) dx_i$ where the $\Delta$ on the right is the usual Laplacian acting on functions.
2. If $X = \mathbb{C}^n$ with its usual flat metric, then $\Delta \partial = \Delta \bar{\partial} = \frac{1}{2} \Delta$.

Exercise 3
Suppose $X$ is Kähler and $\alpha$ is a closed $(1,1)$-form which is primitive (i.e. $\Lambda(\alpha) = 0$). Show that $\Delta \alpha = 0$.

Exercise 4
1. Suppose $V$ is a vector space with compatible inner product, complex structure and fundamental form $(g, I, \omega)$. Suppose $W \subset V$ is a subspace of dimension $2m$. Choose an orientation on $W$; together with $g$ this induces a volume form $\text{vol}_W$. Show that $\text{vol}_W/\omega^m|_W \geq \frac{1}{m!}$, with equality if and only if $W$ is a complex subspace of $V$, i.e. if $IW = W$.
2. Suppose $X$ is a Kähler manifold and $Y$ a compact submanifold of dimension $2m$. Show that $\text{vol}(Y) \geq \int_Y \frac{\omega^m}{m!}$, with equality if and only if $Y$ is a complex submanifold of $X$.
3. Suppose $X$ is a Kähler manifold for which $\omega$ is exact ($\omega = d\alpha$ for some $\alpha$). Show that $X$ has no compact complex submanifolds (in particular $X$ is not compact).

Exercise 5
1. Verify by hand that the Kähler identities hold on $\mathbb{C}^n$.
2. Verify that the Kähler identities do not hold for the Hermitian metric on $\mathbb{C}^2$ with fundamental form $\omega = idz_1 \wedge d\bar{z}_1 + i(|z_1|^2 + 1)dz_2 \wedge d\bar{z}_2$. (For example, try computing $[\partial, L]$.)

Exercise 6
Suppose $X$ is a compact Kähler manifold, of dimension $n$.
1. Show that the Kähler form $\omega$ is harmonic.
2. Show that holomorphic \((n, 0)\)-forms on \(X\) are harmonic, and harmonic \((n, 0)\)-forms are holomorphic. (Note that this means the space of harmonic \((n, 0)\)-forms on \(X\) is actually independent of the Kähler metric we choose.)

3. Show that any two cohomologous Kähler forms \(\omega, \omega'\) are related by \(\omega = \omega' + i\partial\bar{\partial}f\) for some real function \(f\).

**Exercise 7**

A closed 2-form \(\omega\) on a \(C^\infty\) manifold \(M\) of dimension \(2m\) is called a *symplectic structure* if \(\omega\) is closed and nondegenerate at every point.

1. Show that a compact symplectic manifold has \(b_{2k} \geq 1\) for \(0 \leq k \leq m\). (Hint: show that \(\omega^m\) is not exact.)

2. Show that Kähler manifolds carry natural symplectic structures.