Main subject of this course: the differential geometry of complex manifolds. "Manifold" will always mean "$C^\infty$ manifold.

Def: A holomorphic atlas on a manifold is an atlas $\{(U_\alpha, \psi_\alpha)\}$ where each $\psi_\alpha(U_\alpha) \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ and the transition functions $\psi_{\beta} = \psi_{\alpha} \circ \psi^{-1}_{\alpha}$ are holomorphic.

Two atlases $\{(U_\alpha, \psi_\alpha)\}$ and $\{(U'_\beta, \psi'_\beta)\}$ are called equivalent if all $\psi_{\beta} \circ \psi'^{-1}_{\beta}$ are holomorphic.

Def: A complex manifold (of complex dimension $n$) is a manifold (of real dimension $2n$) with an equiv. class of hol. atlas.

Ex: Any orientable real surface admits a complex manifold structure.
Rk: Not known whether $S^6$ admits a complex manifold structure. ($S^{2n}$, all of the $S^{2n}$ and $S^6$) The natural differential geometry of these objects is Kähler geometry:

Def: (vague) A Kähler metric on $X$ is a Riemannian metric for which at every point there exist Riemann normal coords which are holomorphic,

$$g = \sum_{i=1}^{2n} dz_i \otimes d\bar{z}_i + O(||z||^2)$$

Motivations for studying Kähler manifolds:
1) AG: any smooth $\mathbb{C}$ projective variety is Kähler.
2) DG: Kähler is simplest example of "special holonomy" — many problems simplify — e.g. pb of finding Ricci-flat metrics (Einstein 87)
3) SG: Kähler metric is a natural combination of symplectic and complex structures.
4) Physics: Kähler metric is "the geometry of SUSY in d=4".
A first goal:

We will see that compact Kähler manifolds carry a rich linear-algebraic structure: e.g. cohomology decomposes as

\[ H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^p(X) \]  

\[ H^n(X) \cong H^{2n}(X) \]

- A consequence: if \( X \) is compact Kähler then \( b_{2n+1}(X) \) is even \( \forall n \).

This generalizes the fact that if \( X \) is a Riemann surface of genus \( g \) it has \( b_1 = 2g \).

- Indeed, Kähler structure implies a natural way of splitting \( H^n(X, \mathbb{C}) \cong \mathbb{C}^{2g} \)

\[ H^{1,0}(X) \oplus H^{0,1}(X) \cong \mathbb{C}^g \oplus \mathbb{C}^g \]

Moreover, (in this case) this splitting contains the same information as the complex structure on \( X \) — so it’s a linear algebraic way of encoding the \( \mathbb{C} \) geometry!

Other goals: as listed in a short PDF on the course Web page.
(Feedback requested!)

We’ll start with some purely complex-analytic stuff.
(Then, in a few lectures, we’ll see how to get new insights into this stuff using a Kähler metric.)