Consider $X = \mathbb{C}^n$. Write $z_i = x_i + iy_i$; $T(\mathbb{C}^n)$ is real of dim $= 2n$.

Let $\frac{\partial}{\partial z_i} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i} \right)$, then the complexified tangent bundle $T^c \mathbb{C}^n = T \mathbb{C}^n$, similarly $\frac{\partial}{\partial \bar{z}_i}$

Now define an almost $\mathbb{C}$ structure on $T \mathbb{C}^n$ by using $T \mathbb{C}^n = \mathbb{C}^n$, or explicitly

$$I(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}; \quad I = \left\{ \begin{array}{ll} +i & \text{on } T^{1,0} \mathbb{C}^n = \text{Span } \{\frac{\partial}{\partial z_i}\} \\ -i & \text{on } T^{0,1} \mathbb{C}^n = \text{Span } \{\frac{\partial}{\partial \bar{z}_i}\} \end{array} \right.$$

Prop. If $f: \mathbb{C}^n \to \mathbb{C}^n$ then in the basis $\left\{ \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2} \right\}$ for $T \mathbb{C}^n$, $df$ is represented by $[J \quad K \quad 0 \quad 0 \quad 0]$ where $J = \left[ \frac{\partial f_i}{\partial z_j} \right]_{ij}$, $K = \left[ \frac{\partial f_i}{\partial \bar{z}_j} \right]_{ij}$.

Pf. The dual basis is $\{dz_1, dz_2, d\bar{z}_1, d\bar{z}_2\}$.

$$dz_i(df(\frac{\partial}{\partial z_j})) = \frac{\partial f_i}{\partial z_j}, \text{ etc.}$$

Cor. If $f: \mathbb{C}^n \to \mathbb{C}^n$ holomorphic the df is represented by $[J \quad 0 \quad 0 \quad 0 \quad 0]$. In $p^*X$, $df(I_v) = I(df(v))$.

Prop. If $X$ a $\mathbb{C}$ manifold, $TX$ carries natural structure of holomorphic vector bundle.

Pf. Trivialized over the patches $U_\alpha$ of a hol atlas.

Get a complex structure $I_X$ on $TX$ by $I_X = \psi^{-1}_\alpha \circ I_{\psi_\alpha(x)} \circ \psi_\alpha$.

(this is independent of $\alpha$, since $\psi_\beta^{-1} \circ \psi_\alpha = d(\psi_\alpha \circ \psi^{-1}_\beta)$ and hence commutes with $I$, giving $I = \psi^{-1}_\alpha \circ I_{\psi_\alpha(x)} \circ \psi_\alpha = \psi^{-1}_\beta \circ I_{\psi_\beta(w)} \circ \psi_\beta$)

Thus get decomposition $T \mathbb{C} X = T^{1,0} X \oplus T^{0,1} X$.

Moreover, transition functions are $\psi_\alpha \circ \psi_\beta^{-1} = \left[ \begin{array}{cc} J_{\alpha \beta} & 0 \\ 0 & J_{\beta \alpha} \end{array} \right]$ where $J$ is Jacobian of $\psi_\alpha \circ \psi_\beta^{-1}$.

$J_{\alpha \beta}$ is holomorphic $\Rightarrow T^{1,0} X$ has natural holomorphic structure.

Underlying real bundle is $TX$. \hfill \square
Def An almost $C^\infty$ manifold is a pair $(X, I)$ where $X$ is a real manifold and $I$ is a $C^\infty$ section of $\text{Hom}(TX, TX)$ with $I^2 = -1$.

Cor A complex manifold is almost complex.