Consider the trivial $C^0$ line bundle $L \to \Sigma_\tau$.

Sections are just functions on $\Sigma_\tau$, or doubly-periodic functions on $\mathbb{C}$.

Define $\overline{\partial}_E = \overline{\partial} + \alpha \cdot d\overline{z}$, for some $\alpha \in \mathbb{C}$.

This gives a bunch of holomorphic line bundles $L_\alpha$, with $L_\alpha \otimes L_\beta = L_{\alpha + \beta}$.

Prop

$L_\alpha$ is trivial $\iff$ $\alpha = \frac{-2\pi i (m\tau + n)}{\tau - \overline{\tau}}$ for some $m, n \in \mathbb{Z}$

Pf

Global sections? $(\overline{\partial} + \alpha \cdot d\overline{z}) f = 0$

$\implies f = k(z) \cdot e^{-\alpha \overline{z}} \text{ k(z) hol.}$

or better, $f = h(z) \cdot e^{\alpha(z - \overline{z})}$

This requires $h(z + 1) = h(z)$

$h(z + \tau) = h(z) e^{-\alpha(\tau - \overline{\tau})}$

One solution to these constraints is $h(z) = \lambda e^{2\pi i mz}$ for some $m \in \mathbb{Z}$.

then $\frac{h(z + \tau)}{h(z)} = e^{2\pi i m \tau} = e^{-\alpha(\tau - \overline{\tau})}$. So, need $2\pi i (m\tau + n) = -\alpha(\tau - \overline{\tau})$.

Moreover, these are the only solutions, (but this needs some analytic tools! Later in the course)

Re 1) Thus, the line bundles $L_\alpha$ up to isomorphism are parameterized by $\alpha \in \mathbb{C}/(2\pi i \mathbb{Z}) \cong \Sigma_\tau$

2) They form an honest group! (Unlike $\Sigma_\tau$ itself, which when considered as an abstract $\mathbb{C}$ manifold up to $\approx$ is not a group.)

3) Are there all the top trivial line bundles over $\Sigma_\tau$? Yes. Indeed:

Suppose we had a general $\overline{\partial}_E = \overline{\partial} + \beta \in C^0 (\mathcal{S}^1)$

$\overline{\partial}_E' = \overline{\partial} + \beta'$

These give equivalent hol structures if $\beta' - \beta = \frac{\overline{\partial}_E - \overline{\partial}}{g}$ for some $g: \Sigma_\tau \to \mathbb{C}^\times$. So the Q is: can we reduce $\beta$ to a constant by such shifts? An analytic question. (Answer: yes.)

4) We'll see a similar story for line bundles on a general compact Kähler manifold.