Hermite Structures

We discussed real vector spaces \( V \), almost \( \mathbb{C} \) structures \( I: V \to V \).

Now, introduce compatible metrics.

**Def** A positive definite symmetric bilinear form \( g \) on \( V \) is called compatible with \( I \) if \( g(Iv, Iw) = g(v, w) \) \( \forall v, w \in V \) [i.e. \( I \in O(V) \)].

**Def** Given \((V, g, I)\) compatible, define \( \omega: \bigotimes^2 V \to \mathbb{R} \) by \( \omega(v, w) = g(Iv, w) \) ("fundamental form"), and \( h = g - i\omega \).

**Prop** \( \omega \in \Lambda^1 V^* \).

**Pf**
\[
\omega(v, w) = g(Iv, w) = g(IIV, Iw) = -g(v, Iw) = -g(Iv, v) = -\omega(v, v).
\]
\[
\omega(Iv, Iv) = g(IIV, Iv) = g(Iv, w) = \omega(v, v) \Rightarrow \omega \text{ is alternating} \quad V^\otimes V^\otimes \sim V^\otimes V^\otimes V^0.
\]

**Rem** Any two of \((I, g, \omega)\) determine the third.

**Prop** \( h \) is a Hermitian metric on \((V, I)\) viewed as a complex vector space.

**Pf** \( h(v, w) = \overline{h(w, v)} \)

\[
h(Iv, w) = g(Iv, w) - i\omega(Iv, w)
= g(Iv, w) + i g(v, w)
= i (g(v, w) - i\omega(v, w))
= i h(v, w)
\]

There's another Hermitian metric around:

**Def** Extend \( g \) from \( V \) to a Hermitian metric \( g \) on \( V_C \).

**Prop** Under \((V, I) \cong V^\otimes V^\otimes V_C\), \( h \) corresponds to \( 2g \).

**Pf**
\[
h(v, w) = g(v, w) - i g(Iv, w)
= g(v - iIV, w)
= \frac{1}{2} g(v - Iw, w - Iw)
\]
Def \textbf{Grau} \((V, I, \omega)\):
the Lefschetz operator \(L : \Lambda^p(V^*) \to \Lambda^{p+1}(V^*)\) is \(\omega \mapsto \omega \wedge \omega\).

Def Suppose \(V\) oriented, \(\dim V = n\). Let \(vol \in \Lambda^n(V)\) be \(e_1 \wedge \ldots \wedge e_n\) where \(\{e_i\}\) is a positively oriented orthonormal basis.

Then define Hodge \(*\)-operator by \(\omega \mapsto g(\omega, \omega) \cdot \text{vol}\).

Ex \(\star(1) = \text{vol}\): \(\star(e_1) = e_2 \wedge e_3\), \(\star(e_2) = e_3 \wedge e_1\), \(\star(e_3) = e_1 \wedge e_2\), \(\star(e_4) = e_3 \wedge e_2 \wedge e_1\).

Prop \(g(\alpha, \star \beta) = (-1)^{k(n-k)} g(\star \alpha, \beta)\) for \(\omega \in \Lambda^k(V)\).

Prop \(\star^2 = (-1)^{k(n-k)} \cdot \Lambda^k(V)\).

Prop \(\star\) is an isometry.

Pf Exercise.

Def Dual Lefschetz operator \(\Lambda : \Lambda^k(V^*) \to \Lambda^{n-k}(V^*)\) is the adjoint to \(L\) wrt \(g\).

Prop \(\Lambda = \star^{-1} \circ L \circ \star\).

Pf \[g(L\alpha, \beta) \cdot \text{vol} = \Lambda \alpha \wedge \beta = \omega \wedge a \wedge b = a \wedge \omega \wedge b = g(\alpha, \star^{-1} \circ L \circ \star \beta) \cdot \text{vol}\]

Def Extend \(\star\) \(L, \Lambda\)-linearly to \(V^*\). (Warning: some people define \(\star\) to be conjugate-linear.)

Prop \(\alpha \wedge \star \beta = g(\alpha, \beta) \cdot \text{vol}\).

Pf Exercise.

Now say \(V\) has almost \(C\) str. \(I\). So \(n = 2m\), and there's canonical orientation.

(choose a basis for \(V\) as \(C\) vector space and take \(e_1, \ldots, e_m, \ldots, e_{2m}\))
Prop  \[ \ast : \Lambda^p V^* \to \Lambda^{m-p, q-p} V^* \]
\[ L : \Lambda^p V^* \to \Lambda^{p+1, q+1} V^* \]
\[ \Lambda : \Lambda^p V^* \to \Lambda^{p-1, q-1} V^* \]

Pf  Exercise.

Def  \[ H : \Lambda^*(V_0^*) \to \Lambda^*(V_0^*) \quad H=(k-m) \cdot 1 \text{ on } \Lambda^k(V^*) \]
extended $C$-linearly to $V_0^*$

Now, $sl_2 \mathcal{R}$ appears:

Prop  a)  $[H, L] = 2L$
               
    b)  $[H, \Lambda] = -2\Lambda$
               
    c)  $[L, \Lambda] = H$

Pf  a), b) just say $L, \Lambda$ raise/lower the degree by 2.
               
    c) induction on dimension. If $V = V_1 \oplus V_2$ (compatible w/ $I_1, g$) then
\[ L = L_1 \oplus 0 + 0 \oplus L_2 : \Lambda^p V_1 \otimes \Lambda^p V_2 \to \Lambda^{p+1} V_1 \otimes \Lambda^p V_2 \oplus \Lambda^p V_1 \otimes \Lambda^{p+1} V_2 \]
\[ \Lambda = \Lambda_1 \oplus 0 + 0 \oplus \Lambda_2 . \]
\[ H = H_1 \oplus 0 + 0 \oplus H_2 \quad \text{Use this to reduce to case } \dim V = 2. \]

If $\dim(V) = 2$: take an ON-basis $e_1, e_2$ with $Ie_1 = e_2, Ie_2 = -e_1.$

Then $w = e_1 \otimes e_2 \quad [\text{check!}]$
\[ \Lambda^*(V^*) = R \oplus (e_1 R \oplus e_2 R) \oplus w R \]
\[ L : 1 \mapsto w \quad \Lambda : w \mapsto 1 \quad \text{and compute the commutators directly.} \]
Def \[ P^k = \{ \alpha | \lambda(\alpha) = 0 \} \subset \Lambda^k(V^*) \]. (primitive) \[ n = 2m \]

Prop
i) \[ \Lambda^k V^* = \bigoplus_{i \geq 0} L_i(P^{k-2i}) \], orthogonal for \( g \).
ii) \[ k > n \Rightarrow P^k = 0 \].
iii) \[ L^{-k}: P^k \rightarrow \Lambda^{2m-k} V^* \] is injective for \( k \leq m \)
iv) \[ L^{m-k}: \Lambda^k V^* \rightarrow \Lambda^m V^* \] is bijective for \( k \leq m \)
v) \[ \text{If } k \leq m \text{ then } P^k = \{ \alpha | L^{m-k+1} \alpha = 0 \} \].

Pf Basic tool is \( sl_2 \mathbb{R} \) rep theory, which we take as a given:
any 1-d rep is \( \bigoplus \) of irreps; irreps \( W \) labeled by \( r = 0, 1, 2, \ldots \)
\[ W_r = \bigoplus_{l=0} V_{2l} \], \[ H w = (2l-r) w \text{ for } w \in V_{2l} \], \[ \dim_{\mathbb{R}} V_{2l} = 1 \]

\[ H^3 \]
\[ \downarrow \]
\[ H^1 \]
\[ \downarrow \]
\[ H = -1 \]
\[ \uparrow \]
\[ H = -3 \]

So the primitive elements are the ones which are at the bottom; all other elements are obtained by acting with \( L \) on these (proves ii)
they all have \( H \leq 0 \) (proves iii)
etc.

This dump is compatible w/ bidgree dump so e.g. \[ P^k_C = \bigoplus_{p+q=k} P^p \]

Ex \[ n = 2 \]:
\[
\begin{array}{ccc}
1 & 2 & 2 \\
2 & 4 & 1 \\
1 & 2 & 1
\end{array}
\]

\[
\begin{array}{cccc}
\text{W}_0 & \text{W}_1 & \text{W}_2 & \text{W}_3 \\
H=2 & H=1 & H=0 & H=-1 \\
& H=-2
\end{array}
\]

Primitive part:
\[
\begin{array}{ccc}
1 & 3 & 1 \\
2 & 2 & 1
\end{array}
\]