Def. Call \((X, g)\) Kähler if \(X\) is a complex manifold, \(g\) a compatible Hermitian metric, and \(dw = 0\).

**Ex.** \(\dim_c X = 1 \implies \) any Hermitian metric on \(X\) is Kähler.

This is a stronger condition than it first appears!

**Lemma.** \(M\) almost C, \(g\) Hermitian: \(I\) integrable \(\iff\) \(\nabla_{IX} I = I(\nabla_X I)\) \(\forall X, Y\)

\[
N(X,Y) = [X,Y] + I([X,Y] + I[X,Y] - [IX, IY])
\]

\(\text{(if } \nabla X = \nabla Y = 0 \text{ at } p \text{)}\)

\[
= -I(\nabla_Y I)\cdot X + I(\nabla_X I)\cdot Y - (\nabla_Y I)\cdot X + (\nabla_X I)\cdot Y \tag{A}
\]

This shows \((\Leftarrow)\). \(\Rightarrow \) \(\text{For } (\Rightarrow), \text{ let } A(X,Y,Z) = g(\nabla_{IX} I - I(\nabla_X I) Y, Z). \text{ (A) }\implies A(X,Y,Z) = A(Y,X,Z). \text{ But } A(X,Y,Z) = -A(X,Y,Z) \text{ since } I, \nabla \text{ commuting skew-adjoint. Together then say } A = 0. \)

**Prop.** \(g\) Kähler \(\iff\) \(I\) is parallel w.r.t. the Levi-Civita connection \(\nabla\).

\(\Leftarrow\)

\(\omega(v,w) = g(\nabla v, w)\) and \(\nabla g = \nabla I = 0 \implies \nabla \omega = 0\)

\(\iff\) Set \(B(X,Y,Z) = g(\nabla_{IX} I, Y, Z). \text{ Then } B(X,Y,Z) = B(X,Y,Z)\) \(\text{and Lemma says } B(X,Y,Z) + B(IX,Y,Z) = A = 0. \)

Now, \(0 = \text{ dw (X,Y,Z) = } (\nabla_w \omega)(Y,Z) + (\nabla_v \omega)(Z,X) + (\nabla_z \omega)(X,Y) \quad [\nabla \text{ torsion-free}]\)

\(B(X,Y,Z) + B(Y,Z,X) + B(IX,Y,Z) = 0 \quad [\nabla g = 0]\)

\[
\implies \begin{cases} \quad B(X,Y,IZ) + B(Y,IZ,X) + B(IZ,X,Y) = 0 \\
B(X,Y,Z) + B(IX,Y,Z) + B(Z,X,Y) = 0
\end{cases}
\]

Add these, we \((\Leftarrow) \implies 2B(X,Y,IZ) = 0, \text{ i.e. } B = 0, \text{ as desired}\)

**Rk.** A Kähler metric involves \((I, \omega)\) which each obey an integrability condition, which are compatible \((\omega \in \Omega^1, \text{ i.e } \omega(\nabla v, w) = \omega(v,w))\)

and obey a positivity condition:

\[
\text{(equiv. } \omega = \frac{i}{2} h_{ij} dz_i \wedge d\bar{z}_j \text{ with } h + \text{def})
\]

**Prop.** \(X\) complex \(\Rightarrow\)

\(\{\text{Kähler metrics on } X\} \leftrightarrow \{\omega \in \Omega^1(X, \mathbb{C}) \mid g(X,Y) = \omega(X, IY) + \omega(IX, Y) \text{ pos def, and } dw = 0\}\)

Thus there is a cone of allowed Kähler forms on a given complex manifold \(X\).
One way to produce a closed $$(1,1)$$-form: write \(\omega = i \overline{\partial} \partial K \) for \(K \in \Omega^0, \text{real}\).

(In fact every \(\omega\) is locally of this form — will prove later.)

**Ex** \(\mathbb{C}^n\) is Kähler w/ usual flat metric, Kähler potential \(K = \sum_{i=1}^{n} \frac{1}{i^2} |z_i|^2\).

But usually, \(K\) doesn't exist globally. Rather, have different \(K\) in different patches, related by \(K' = K + \text{Re}(f(z))\) for \(f\) holomorphic.

(then \(\overline{\partial} K' = \overline{\partial} K\), so \(\omega\) still globally defined!)

**Ex** Say \(X = \mathbb{CP}^n\). Morally, say \(K = \frac{1}{2\pi} \log \|w\|^2 = \frac{1}{2\pi} \log (\sum_{j=0}^{n} |w_j|^2)\)

This isn't really a well defined function. But, it's OK:

change \(w_i \rightarrow f w_i\), shift \(K \rightarrow K + \frac{1}{2\pi} \log |f|^2 = K + \text{Re}(\frac{1}{2\pi} \log f)\).

More precisely: take \(K_i = \frac{1}{2\pi} \log (1 + \sum_{i+j=n} |z_i|^2)\) on patch \(U_i\).

Then define \(\omega = i \overline{\partial} \partial K_i\); this is well defined and closed.

Just need to check positivity: in the patch \(U_0\), say, use coords \(z_1, \ldots, z_n\),

\[
\omega = \frac{1}{(1+\sum |z_i|^2)^2} \sum h_{ij} dz_i \wedge d\overline{z_j}
\]

with \(h_{ij} = (1+\sum |z_i|^2) \delta_{ij} - \overline{z_i} z_j\)

so \(h(w,\overline{w}) = h_{ij} w_i \overline{w_j} = (1+\sum |z_i|^2) \|w\|^2 - |(z,\overline{w})|^2 > \|z\|^2 \|w\|^2 - |(z,\overline{w})|^2 > 0\) [by Cauchy-Schwarz]

Thus we get a Kähler metric on \(\mathbb{CP}^n\) [Fubini-Study]

**Rk** This Kähler structure is not canonical on \(\text{IP}(V)\) — depends on a choice of Hermitian metric in \(V\)!
Our normalization was chosen so that $\frac{1}{C^{IP^1}} \int_{\mathbb{C}^{IP^1}} \frac{i}{2\pi} \frac{1}{1+|z|^2} \, dz \wedge d\bar{z} = 2 \int_0^\infty \frac{r \, dr}{(1+r^2)^2} = 1.$

$\text{Ex}$: Torus $\mathbb{C}^n/\Lambda$ is Kähler ($\Lambda$ any lattice).

$\text{Prop}$: If $X$ is Kähler, any complex submanifold $Y \subset X$ is also Kähler.

If $\text{Just take the restriction of all the structures:}$

$I_Y = I_X \big|_{TY}, \quad g_Y = g_X \big|_{TY}, \quad \omega_Y = i^* \omega_X \quad (\ni: Y \hookrightarrow X)$

Compatible on $X \Rightarrow$ compatible on $Y$ ✅

$\text{Cor}$: Any smooth projective algebraic variety is Kähler.

"Geometric" interpretation of $K$:

**Def**: A Hermitian metric $h$ on a $C^\infty$ bundle $E \to X$ is a Hermitian bilinear form $h_x$ on each fiber $E_x$, varying smoothly.

Suppose given a hol. l.b. $L \to X$ w/ a Hermitian metric.

Choose local section $s_\alpha \in \Gamma(U_\alpha, L)$ over patch $U_\alpha$, define $K_\alpha$ by

$h(s_\alpha, s_\alpha) = e^{K_\alpha}$. On overlaps, $s_\beta = T_{\alpha\beta} s_\alpha$ gives $K_\beta = K_\alpha + \log |T_{\alpha\beta}|^2$

Thus, $\omega = i \partial \bar{\partial} K_\alpha$ makes global sense, $\alpha$-indep. $K_\alpha$ came from the globally defined $h$ on $L$.

[In fact, the $K$ we used on $C^{IP^1}$ arose this way, with $L = \mathcal{O}(-1)$, and Hermitian metric induced from the standard one on $C^{n+1}$]