The $p$-form Laplacian

On an oriented Riem manifold:

$$d: \Omega^p(M) \to \Omega^{p+1}(M)$$

**Def**

1. $d^*: \Omega^p(M) \to \Omega^{p-1}(M)$ given by $d^* \omega = (-1)^{n(p+1)+1} \ast d^* \omega$

2. $\Delta: \Omega^p(M) \to \Omega^p(M)$ given by $\Delta = dd^* + d^* d$

**Prop**

If $M$ compact, for $L^2$ pairing $\langle \alpha, d^* \beta \rangle_{L^2} = \langle d\alpha, \beta \rangle_{L^2}$

**Pf**

$$\langle \alpha, d^* \beta \rangle_{L^2} = \int \langle \alpha, d^* \beta \rangle \text{ vol} = \int \alpha \wedge ^* \beta$$

$$= \langle d\alpha, \beta \rangle_{L^2} = \int d\alpha \wedge ^* \beta$$

$$= (-1)^{n+|\alpha|} \int \alpha \wedge d^* \beta$$

$$= (-1)^{|\alpha| + |\alpha|(n-|\alpha|)} \int \alpha \wedge ^*(d^* \beta)$$

$$= \langle \alpha, d^* \beta \rangle_{L^2} \quad \text{(since } |\alpha| + |\alpha|(n-|\alpha|) = n(|\beta|+1) + 1 \text{ mod } 2\text{)}$$

Thus we call $d^*$ a "formal adjoint" to $d$.

**Cor**

If $M$ compact, $\langle \alpha, \Delta \beta \rangle_{L^2} = \langle d\alpha, d\beta \rangle_{L^2} + \langle d^* \alpha, d^* \beta \rangle_{L^2} = \langle \Delta \alpha, \beta \rangle_{L^2}$

**Cor**

If $M$ compact, $\Delta \alpha = \lambda \alpha$, then $\lambda \geq 0$; if $\lambda = 0$ then $d\alpha = 0$, $d^* \alpha = 0$.

**Pf**

$$||\alpha||_{L^2}^2 = \langle \alpha, \Delta \alpha \rangle_{L^2} = ||d\alpha||_{L^2}^2 + ||d^* \alpha||_{L^2}^2$$

**Rk**

This really needs $M$ compact — e.g. if $M = \mathbb{R}$ and $f(x) = e^x$, $\Delta f = -f$.

**Def**

$H^p(M) = \ker (\Delta: \Omega^p(M) \to \Omega^p(M))$

**Cor**

$\dim H^0(M) = \# \text{ connected components of } M = b^0(M)$. 
This fact has an important refinement:

**Def (de Rham cohomology) M smooth manifold:** \( H^p_{\text{dR}}(M) = \frac{\ker (d: \Omega^p(M) \to \Omega^{p+1}(M))}{\text{im}(d: \Omega^{p+1}(M) \to \Omega^p(M))} \)

\[ b^p(M) = \dim_{\mathbb{R}} H^p_{\text{dR}}(M) \]

So this is another way of thinking about the "usual" cohomology of \( M \).

**Thm (Hodge) If** \( M \) **compact Riemannian,**

Then each class in \( H^p_{\text{dR}}(M) \) contains a unique element of \( H^p(M) \).

**Rk Note** \( H^p_{\text{dR}}(M) \) **is defined without a metric,** while \( H^p(M) \) **depends on one a priori.**

**Pf Sketch** If \( \omega \in H^p(M) \) then \( d\omega = 0 \), so have a map \( H^p(M) \to H^p_{\text{dR}}(M) \).

- **Injective:** suppose \( \omega \in H^p(M) \), \( \omega = d\alpha \); then \( \|\omega\|^2 = \langle \omega, d\alpha \rangle = \langle d\ast\omega, \alpha \rangle = 0 \).

- **Surjective:** first note \( \text{Im} d, \text{Im} d\ast, \) and \( H^p \) are all mutually orthogonal.

Suppose we knew \( \Omega^p = d\Omega^{p-1} \oplus d\ast\Omega^{p+1} \oplus H^p \). (see below)

Then, given \( \gamma \) with \( d\gamma = 0 \), \( \gamma = d\alpha + d\ast\beta + \delta \)

\[ d\delta = d\ast\beta = 0 \]

but then \( \langle \beta, d\ast\beta \rangle_{L^2} = 0 \) so \( \|d\ast\beta\|^2 = 0 \), i.e. \( d\ast\beta = 0 \).

So, \( \gamma = d\alpha + \delta \).

But then \( [\gamma] = [\delta] \) in \( H^p \).
So, what we need is to prove

**Lemma** \( \Omega^p = d \Omega^{p-1} \otimes d^* \Omega^{p+1} \otimes H^p \).

**Pf. Sketch** It would be enough to show \( \Omega^p = \Delta \Omega^p \otimes H^p \).

(since \( \text{Im} \Delta \subset \text{Im} d \otimes \text{Im} d^* \))

Note: this would be easy in finite-dimensional setting: just diagonalize \( \Delta \)

to see \( \exists G : \Omega^p \rightarrow \Omega^p \) s.t. for \( \omega \in (H^p)^\perp \), \( \Delta^* G \omega = \omega \).

(So in pth \( \omega \in \text{Im} \Delta \).)

In infinite-dimensional setting, need to develop theory of "ellipticity" to show

that \( G \) exists. (This theory also shows that \( H^p \) is finite-dimensional.)

Very rough idea: on \( \mathbb{S}^n \), write \( \Delta f = \sum \frac{\partial^2 f}{\partial x_i^2} \), then \( \tilde{\Delta} f(k) = \|k\|^2 \hat{f}(k) \) \( k \in \mathbb{Z}^n \)

thus \( \tilde{G} f = \frac{f}{\|k\|^2} \). No problem, as long as \( \hat{f}(0) = 0 \). A version of this idea really works.

It depends crucially on \( \|k\|^2 \neq 0 \) when \( k \neq 0 \). This is ellipticity of \( \Delta \).

\( \textbf{Rk.} \) Warning: \( \alpha, \beta \) harmonic \( \Rightarrow \alpha \wedge \beta \) harmonic!

So \( \wedge \) does not reproduce the "cup product."