Def: \(Y \subset X\) is a hypersurface if it is locally cut out by an equation \(f_i = 0\), \(f_i\) holomorphic on \(U_i\).

But not globally, since there might not even be nontrivial global hol \(f_i^\ast\) on \(X\)!
Still, something analogous is true: we could take the local \(f_i^\ast\)'s and stitch them together into a line bundle —

\[\text{Def/Prop } \mathcal{O}(\mathcal{Y}) \text{ is the sheaf on } X \text{ whose local sections are hol functions on } X \text{ which vanish on } \mathcal{Y}.\]

\[\text{Def/Prop } \mathcal{O}(\mathcal{Y}) \text{ is the sheaf of sections of a line bundle over } X, \text{ which we also call } \mathcal{O}(\mathcal{Y}).\]

\[\text{Pf } \text{It's a locally free sheaf of } \mathcal{O}-\text{modules over } X.\]
\(\) (Generated by \(f_i^\ast\) on \(U_i\)). Explicitly: transition funcs.
\[\varphi_{ij} = \frac{f_i}{f_j}.\]

\[\text{NB: } \mathcal{O}(\mathcal{Y}) \text{ is generally different from } \mathcal{O} \text{ even in the } C^\infty \text{ sense!}\]

\[\text{e.g. } \mathcal{Y} \subset \mathbb{C}^2, \quad U_i = \{|z| > \varepsilon\}, \quad U_j = \{|z| < 2\varepsilon\}, \quad \varphi_{ij} = \frac{z^2}{1} = z.\]

\[\text{has nontrivial vanishing at } \mathcal{Y}.\]

\[\text{as a map } U_{ij} \to \mathbb{C}^x.\]

Def: \(\mathcal{O}(Y) = \mathcal{O}(\mathcal{Y})^\ast\).

Prop: \(\mathcal{O}(Y)\) admits a canonical global section \(s_\mathcal{Y}\) whose zero locus is exactly \(Y\).

Pf: \(s_\mathcal{Y}\) is the operation of evaluation, acting on sections of \(\mathcal{O}(\mathcal{Y})\).
\(\) It clearly vanishes on the fibers of \(\mathcal{O}(\mathcal{Y})\) over \(Y\),
\(\) doesn't vanish elsewhere.

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\[\text{Def } \mathcal{O}(Y) = \mathcal{O}(\mathcal{Y})^\ast.\]
Def A divisor $D$ is a formal $\Sigma$ of hypersurfaces, with $\mathbb{Z}$-weights,
$$D = \sum_{Y} n_Y [Y] \quad n_Y \in \mathbb{Z} \quad \text{(locally finite)}$$
Divisors form an abelian group in the obvious way: $n[Y] + m[Y] = (n+m)[Y]$.

Def \(\mathcal{O}(D) = \bigotimes_{Y} \mathcal{O}(Y)^{\otimes n_Y}\).

Def Say $D \sim D'$, "$D$ is linearly equivalent to $D'$", if $\mathcal{O}(D) \cong \mathcal{O}(D')$.

Def If $f \in K(X)$ (meromorphic function)
- $\nu_Y(f) =$ order of vanishing of $f$ along $Y$ (defined locally by $f = f_Y^{\nu_Y(f)} g$ with $g \in \mathcal{O}^X$ — positive if $f$ has a zero, negative if $f$ has a pole — warning: need some work to be sure this really defines $\nu_Y(f)$ well defined globally)

- $\text{div}(f) = \sum_{Y} \nu_Y(f) [Y]$ (sum over irreducible hypersurfaces)

Prop $D \sim D' \iff \exists f \in K(X)$ with $\text{div}(f) = D - D'$
$$\iff \exists s \in \mathcal{O}(D) \text{ with } \text{div}(s) = D'.$$

Pf Exercise.

For $X$ a curve, divisors are just sets of points weighted by integers, so

\underline{Abel-Jacobi map} \quad X \rightarrow \text{Pic}(X)
$$z \rightarrow \mathcal{O}(z)$$

\text{(Exercise: Injective for $X$ of genus $>0$, \approx \text{ for $X$ of genus 1 }.)}$
\text{(More generally have maps $X^d \rightarrow \text{Pic}(X)\ldots$)}
Prop. \( \text{div}(fg) = \text{div}(f) + \text{div}(g) \).

Pf. Easy.

Def. Call \( D = \sum n_y [Y] \) effective if all \( n_y \geq 0 \).

Ex. On \( \mathbb{P}^1 \), \( D = \sum n_i [z_i] \) has \( O(D) \sim O(\sum n_i) \).

E.g. \( O([z_1] - [z_2]) \sim O \), because the function \( f(z) = \frac{z - z_1}{z - z_2} \)

has \( \text{div}(f) = [z_1] - [z_2] \).

equivalently, \( O([z_1]) \sim O([z_2]) \sim O(1) \).

But on torus, \( O(p_1) \not\sim O(p_2) \).

Normal bundles

\[ Y \subset X \text{ complex submanifold}: \quad \text{define } NY = \frac{T_{\text{hol}}X}{T_{\text{hol}}Y} \quad \text{(hol. bundles)} \]

\[ 0 \to T_{\text{hol}}Y \to T_{\text{hol}}X \vert_Y \to NY \to 0 \]

Prop. If \( Y \) is a (smooth) hypersurface, then \( NY \sim O(Y) \vert_Y \).

Pf. Say \( Y \) cut out locally by \( f_i = 0 \)

\( \text{df}_i \) is a local section of \( (NY)^* \subset (T_{\text{hol}}X \vert_y)^* \). \( Y \) smooth \( \Rightarrow \) \( \text{df}_i \neq 0 \).

Then \( \text{df}_i = d(\Psi_{ij} f_j) \quad \left[ \Psi_{ij} = \frac{f_i}{f_j} \text{ nonvanishing regular} \right] \)

\[ = d\Psi_{ij} f_j + \Psi_{ij} df_j \]

\[ = \Psi_{ij} df_j \text{ on } Y \]

Thus we have a well def. isom \( (NY)^* \sim O(-Y) \vert_Y \)

\( \text{df}_i \mapsto f_i \)