Calculating sheaf cohomology

Given an open cover $\mathcal{U} = \{U_i\}$ of $M$ we defined the Čech resolution $C^\cdot(\mathcal{U})$ of any sheaf $F$ over $M$. Recall $U = \bigcap_{i \in I} U_i$.

**Prop** Suppose $H^p(U_i, F) = 0 \forall p > 0, I$. [Leray]

Then $H^p(M, F) = H^p(C^\cdot(\mathcal{U})(F))$

**Pf** By “dimension-shifting” — see [Leray]

[In fact, one can also show on paracompact $M$ that you can always pass to a fine enough cover (Godement) Could define $H^p(M, F)$ this way.]

How to produce good covers?

**Lemma** $U$ contractible $\Rightarrow H^p(U, G) = 0 \quad p > 0$

**Pf** Use resolution by singular cochains. (Is there a more direct way?)

**Ex** $M = S^1$

- $U_1$
- $U_2$

$H^0(M, \mathbb{Z}) = \text{Ker}(d) \cong \mathbb{Z}$

$H^1(M, \mathbb{Z}) = C^1(U, \mathbb{Z}) \cong \mathbb{Z}$

$H^p(M, \mathbb{Z}) = 0$ for $p > 1$

**Ex** Similarly $H^p(C^\cdot, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } p = 0, 1 \\ 0 & \text{for } p > 1 \end{cases}$

**Ex** Calculate $H^\cdot(M, \mathbb{Z})$:

$\begin{array}{ccc}
0 & \rightarrow & C^0(U, \mathbb{Z}) \xrightarrow{d} C^1(U, \mathbb{Z}) \rightarrow 0 \\
& & \approx \mathbb{Z}(U) \oplus \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_1 \cap U_2) \\
& & (a, b) \mapsto (a+b, a-b) \\
\end{array}$

[Remarke: Can also use just one “open set”]

[entire cover]
How about analytic sheaves?

Def. A polydisc in $\mathbb{C}^n$ is $\{z : |z_i| \leq \varepsilon_i\}$ where each $\varepsilon_i \in \mathbb{R}_0^+ \{0\}$.

Ex. $X = \text{polydisc} \Rightarrow H^p(X, \mathcal{O}) = 0 \; \forall p > 0$  
(Because $H^p(X, \mathcal{O}) \cong H^0(X^p, \mathcal{O})$, which vanishes by $\overline{\partial}$-Poincaré lemma)

Ex. $X = \text{polydisc} \Rightarrow H^p(X, \mathcal{O}^{-X}) = 0 \; \forall p > 0$  
(Use $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^{-X} \to 0$)

This admits a massive, important generalization:

Thm. ("Cartan's Theorem B") If $X$ is a Stein manifold (a closed $\mathbb{C}$ submanifold) and $\mathcal{F}$ a coherent sheaf then $H^p(X, \mathcal{F}) = 0 \; \forall p > 0$.

Ex. $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathcal{O}^x \overset{\cdot 2}{\to} \mathcal{O}^x \to 0$  
$\mathcal{O}^x$ is $M = \mathbb{C}^x$  
$\psi(f) = f^2$

$H^0(\mathbb{C}, \mathbb{Z}/2\mathbb{Z}) \to H^0(\mathbb{C}, \mathcal{O}^x) \to H^0(\mathbb{C}, \mathcal{O}^x) \overset{\delta}{\to} H^1(\mathbb{C}, \mathbb{Z}/2\mathbb{Z})$

What is $\delta$ concretely?