Double complexes

We showed that given 2 acyclic resolutions of a sheaf $F$

\[ 0 \to F \to A' \to \cdots \]
\[ 0 \to F \to B' \to \cdots \]

we get

\[ H^i(M, F) \cong H^i(A'(M)) \cong H^i(B'(M)) \]

But how to compare them concretely?

One way: build a double complex of sheaves,

\[ \begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 \\
\downarrow & & & \downarrow & & & \\
0 & F & A^0 & A^1 & A^2 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & B^0 & C^{0,0} & C^{0,1} & C^{0,2} & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & B^1 & C^{1,0} & C^{1,1} & \cdots \\
\downarrow & \downarrow & \downarrow & & & \\
0 & B^2 & C^{2,0} & \cdots & & \\
& & & & & & \\
\end{array} \]

\[ d^2 = 0 \]
\[ \delta^2 = 0 \]
\[ d\delta = \delta d \] (commutativity diagram)

Associated single complex: \[ Z^k = \bigoplus_{p+q=k} C^{p,q} \] with differential \[ D = d + (-1)^p \delta \]

The inclusions \( i_A, i_B \) give natural maps \( i_A : H^i(A'(M)) \to H^i(Z'(M)) \)
\( i_B : H^i(B'(M)) \to H^i(Z'(M)) \)
Prop. If all rows are acyclic resolutions, \( i_B \) is \( \cong \).
   - If all columns \( \cong \) \( i_A \) is \( \cong \).

Pf. 

Voisin, p. 188 (Lemma 8.5)

Best possible case: all rows and cols acyclic.
In that case, can use \( C \) to construct concrete iso. \( H^\ast(A^\ast(M)) \cong H^\ast(B^\ast(M)) \).
(And, claim: it's the same iso. that comes from identifying both with \( H^\ast(M, F) \). Why?)

For example: given \( b^2 \in B^2(M) \) with \( \delta b^2 = 0 \implies [b^2] \in H^2(B^\ast(M)) \)

\[ l_B(b) = c^{2,0} \in C^{2,0} \text{ with } \delta c^{2,0} = 0, \quad d c^{3,0} = 0 \implies [c^{2,0}] \in H^2(C^\ast(M)) \]
so \( \exists c_{1,0} \in C_{1,0} \text{ with } \delta c_{1,0} = c^{2,0} \) (cols acyclic)
then \( d c_{1,0} = c_{1,1} \in C_{1,1} \text{ with } \delta c_{1,1} = 0, \quad d c^{3,0} = 0 \implies [c_{1,1}] \in H^2(C^\ast(M)) \)

and \( D c_{1,0} = c_{1,1} - c^{2,0} \implies [c^{2,0}] = [c_{1,1}] \)

Combining: \( \exists c_{0,1} \in C_{0,1} \text{ with } \delta c_{0,1} = c_{1,1} \) (cols acyclic) then \( d c_{0,1} = c_{0,2} \in C_{0,2} \text{ with } \delta c_{0,2} = 0, \quad d c_{1,1} = 0 \implies [c_{0,2}] \in H^2(C^\ast(M)) \)

and \( D c_{0,1} = c_{1,1} + c_{0,2} \implies [c_{0,2}] = -[c_{1,1}] \)

Finally, \( \exists a^2 \in A^2 \text{ with } i_A(a^2) = c_{0,2} \)

Altogether, \( i_B([b^2]) = [c^{2,0}] = [c_{1,1}] = -[c_{0,2}] = -i_A([a^2]) \)

This "star-step" procedure gives the desired explicit way of passing from a representative \([b^2]\) in one realization of cohomology to its counterpart \([-a^2]\) in another realization. (We use it later in the course.)
Of course, this depends on being able to produce the needed interpolating double complex!

One neat example arises when one of the two resolvents, say $\mathcal{A}^i$, is Čech.

Then there's a natural way to build double complex: just take Čech resolutions of all the sheaves in the resolution $\mathcal{B}^i$.

i.e. $\mathcal{C}^{p,q} = \mathcal{C}^p(\mathcal{B}^q)$