Holomorphic line bundles, redux

Let \( \text{Pic}(X) \) be the group of holomorphic line bundles over \( X \), with multi. given by \( \otimes \).

**Prop** \( \text{Pic}(X) \cong H^1(X, \mathcal{O}^*) \).

**Pf Sketch** Fix a covering \( U \) of \( X \) by polydiscs. \((\exists ?)\)

1. Use description via transition functions to get a map \( H^1(X, \mathcal{O}^*) \to \text{Pic}(X) \). Well defined by change by Čech coboundary doesn't change the line bundle. Injective by \( \mathcal{O} \).
2. Surjective by every line bundle over polydisc is trivial. (prove using characters via \( \widetilde{\Omega} \).

Use \( 0 \to \mathcal{O} \to \mathcal{O} \to \mathcal{O}^* \to 0 \)

to get \( \cdots \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \to \text{Pic}(X) \xrightarrow{\zeta} H^2(X, \mathcal{O}) \to \cdots \).

Thus, loosely speaking, \( \text{Pic}(X) \) has a "continuous" part in \( H^1(X, \mathcal{O}) \cong H^{0,1}(X) \) plus a "discrete part" in \( H^2(X, \mathcal{O}) \). In \( \mathbb{p} \), \( H^{0,1}(X) \) surjects onto sub-group \( \text{Jac}(X) = \text{Pic}^0(X) \) with \( \zeta_1(z) = 0 \). The kernel of \( H^{0,1}(X) \to \text{Jac}(X) \) is \( \text{Im}(H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*)) \). That's injective when \( X \) compact.

To see this, go back a few steps to \( H^{0,1}(X, \mathcal{O}) \to H^{0,1}(X, \mathcal{O}^*) \) which is surjective since we can always take the log of a constant function.

Thus we proved

**Prop** \( X \text{ compact} \Rightarrow \text{Jac}(X) \cong \frac{H^{0,1}(X)}{H^1(X, \mathcal{O})} \).