Def \( \nabla \) any connection in a bundle \( V \rightarrow M \), \( x \in M \):

Parallel transports \( P_{x_1}^{x_2} \in \text{Hom}(V_{x_1}, V_{x_2}) \)

\[
\text{Hol}_\nabla (x) = \{ P_{x_1}^{x_2} : P \text{ preserves smooth paths in } M \text{ from } x_1 \text{ to } x \} < \text{GL}(V_x)
\]

\[
\text{Hol}^\circ_\nabla (x) = \{ \overbrace{\text{null-homotopy}} \}
\]

\( \text{Hol}_\nabla (x) \) is a Lie subgroup of \( \text{GL}(V_x) \). \( \text{Hol}_\nabla (x) \cong \text{Hol}^\circ_\nabla (x) \).

\( \nabla \) flat \( \iff \text{Hol}^\circ_\nabla (x) \) trivial.

Prop (Holonomy principle) \( M \) connected, \( \nabla \) connection in \( V \rightarrow M \), \( x \in M \), \( v \in V_x \):

\( v \in \text{Hol}_\nabla (x) \iff v \) can be extended to \( \nabla \) flat section of \( V \) over \( M \).

Prop (\( \Rightarrow \)) use \( P \) to build the extension:

\[
\text{hol}(y) = IP_{y}^{x} (v) = IP_{y}^{y} \circ IP_{y}^{-1} \circ IP_{y}^{x} (v) = IP_{y}^{x} (v)
\]

Prop (\( \Leftarrow \)) easy.

Ex.

- If \( \nabla \) is the Levi-Civita connection for some Riem. metric, \( \nabla g = 0 \Rightarrow \text{Hol}_\nabla (x) \) acts on \( T^*_x T^* \) preserves \( g \). Equivalently, \( \text{Hol}_\nabla (x) \subset O(T_x) \subset \text{GL}(T_x) \).

- If \( X \) is Kähler and \( \nabla \) is Levi-Civita, \( \nabla g = \nabla \omega = 0 \Rightarrow \nabla h = 0 \Rightarrow \text{then } \text{Hol}_\nabla (x) \subset U(T_x^{1,0}) \subset O(T_x X) \subset \text{GL}(T_x) \)

This is an instance of "special holonomy".

The possible \( \sim \) types of \( \text{Hol}_\nabla (x) \) for \( \nabla \) Levi-Civita have been classified as long as \( M \) is irreducible (not locally a direct \( T \) ) and also not locally a symmetric space, the only possibilities are [Berger, Simms]:
Riemannian $O(n)$

Kähler $U(\frac{n}{2})$

Calabi-Yau $SU(\frac{n}{2})$

Hyperkähler $Sp(\frac{n}{4})$

$G_2$, $Sp_{10}(7)$

Quaternion-Kähler $Sp(\frac{n}{4}) \cdot Sp(1)$