Strominger-Yau-Zaslow picture of CY manifolds

Recall:

**Def**: $X$ symplectic, $L \subset X$ submanifold:

$L$ is Lagrangian if $i^*\omega = 0$ and $\dim L = \frac{1}{2} n$.

**If** $X$ is CY then $X$ has $\Omega \in H^1(K)$ nonvanishing.

**Def**: $X$ CY, $L \subset X$ submanifold:

$L$ is special Lagrangian with phase $\Theta$ if $L$ is Lagrangian and

$$i^*\left[\Im(e^{i\Theta}\Omega)\right] = 0 \quad (i: L \hookrightarrow X)$$

**Rk Prototype**: $X = C^n$, $\omega = i\sum dz_i \wedge \overline{dz}_i$, $\Theta = 0$, $\Omega = dz_1 \wedge \ldots \wedge dz_n$, $L = \mathbb{R}^n \subset C^n$.

**NB**: in this case $i^*\left[\Re(e^{i\Theta}\Omega)\right] = \text{vol}_L$.

More generally: suppose we normalize $|\Omega \wedge \overline{\Omega}| = \text{vol}$.

Then along any submanifold $L$ of dimension $\frac{1}{2} \dim X$ we have

$$|\Re(e^{i\Theta}\Omega)| \leq \text{vol}_L.$$  [Honey-Lawson]

So $\text{vol}(L) \geq \int_L |\Re(e^{i\Theta}\Omega)|$  (free, optimizing over $\Theta$, $\text{vol}(L) \geq \int_L \Omega$.)

Special Lagrangian $L$ saturate this inequality (so in particular they are volume-minimizing: any $L'$ with $[L] = [L']$ has $\text{vol}(L') \geq \text{vol}(L)$).

**cf**: the Kähler form on any Kähler $X$: $dw = 0$, $\frac{\omega^n}{n!} \leq \text{vol}$.

So every $Y$ has $\text{vol}(Y) \geq \int_Y \frac{\omega^n}{n!}$.

Holomorphic $Y$ saturate this inequality.

**Strominger-Yau-Zaslow’s proposal**: every Calabi-Yau manifold $X$ has

$$\pi: X \to S^n$$

where a generic fiber $\pi^{-1}(u)$ is a special Lagrangian $n$-torus.
this structure gives a useful picture of what Ricci-flat Kähler metrics on X actually "look like": as one deforms the complex structure of X to a "large complex structure point," while holding the diameter of X fixed, the metric on X collapses to a metric on $S^n$.

What do we mean by "collapsing"? Gromov-Hausdorff sense:

**Def** If $X,Y$ are metric spaces and $\exists f: X \to Y$, $g: Y \to X$
(not nec. cts) with $|d(x_1x_2) - d(f(x_1), f(x_2))| < \varepsilon$ \ $\forall x_1, x_2 \in X$
and $|d(y, gof(x))| < \varepsilon$ \ $\forall x \in X$

and similarly with $X$ and $Y$ reversed,

then say $d_{GH}(X,Y) \leq \varepsilon$.

Define $d_{GH}(X,Y)$ to be infimum of all such $\varepsilon$.

In dimension $n=1$, we can really understand what this means:

Consider the torus $X = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}_\tau)$

Recall inequivalent $X_\tau$ are labeled by $\tau \in \mathbb{H}/\text{SL}(2,\mathbb{Z})$.

Only one infinite "end": $\tau \to i \infty$.

This will be the "large $\mathbb{C}$ structure" limit.

Say $\tau = i s$, $s \in \mathbb{R}_+$, $z = x + iy$.

Ricci-flat Kähler metric 

has diameter 1 (for $s \geq 1$)

Think of it as a circle fibered over a circle. As $s \to \infty$, fibers collapse, metric approaches a circle in G-H sense.
How about \( n=2 \)? For 4-dim, similar story to \( n=1 \).

For K3, more interesting.

First, let's construct the desired torus fibrations over \( S^2 \).

Do it complex-analytically (even though the fibers were supposed to be Lagrangian: we'll later see that they are, but in a different \( \mathbb{C} \text{st} \)).

So, view \( S^2 \) as \( \mathbb{CP}^1 \). Over each point \( u \in \mathbb{CP}^1 \), we want to put a torus (genus 1 curve). Try representing that as a cubic curve,

\[
\Sigma_u = \{ y^2 z = x^3 + A(u) x z^2 + B(u) z^3 \} \subset \mathbb{CP}^2.
\]

\( X \) will be the union of the \( \Sigma_u \).

What should \( A(u) \) and \( B(u) \) be? Let's try to get some intuition. We want \( X \) to be a K3 surface, so \( \chi(X) = 24 \).

Generic \( \Sigma_u \) is a smooth genus 1 curve \( \Rightarrow \) contributes 0 to \( \chi(X) \).

At a simple zero of \( \Delta(u) = 27 B(u)^2 - 4 A(u)^3 \), \( \Sigma_u \) is singular.

\([y^2 = P_3(x) \text{ where } P \text{ has 1 double, 1 simple root}]\) Let \( D = \text{div}(\Delta) \).

Each such singular fiber has \( X=1 \) \( \Rightarrow \) contributes 1 to \( \chi(X) \).

So, we want \( |D| = 24 \), i.e. globally \( \Delta(u) \) a sec. of \( \mathcal{O}(24) \). Easiest way to get that:

\( B(u) \) a section of \( \mathcal{O}(8) \), \( A(u) \) a section of \( \mathcal{O}(12) \).

So:

Write \( X = \{ y^2 z = x^3 + A(u) x z^2 + B(u) z^3 \} \subset \left[ \mathcal{O}(4) \oplus \mathcal{O}(6) \oplus \mathcal{O} \right] \big/ \mathbb{C}^x \)

where \( A(u) \) is a section of \( \mathcal{O}(8) \)

\( B(u) \) is a section of \( \mathcal{O}(12) \)

\( x \) is valued in \( \mathcal{O}(4) \)

\( y \) is valued in \( \mathcal{O}(6) \)

\( z \) is valued in \( \mathcal{O}(0) \)

and \( \mathbb{C}^x \) acts by \( (x,y,z) \rightarrow (\lambda x, \lambda y, \lambda z) \).
$\mathbb{CP}^4$ has $K \simeq \mathcal{O}(-2)$, so there's a nonvanishing global section $\Omega$ of $K \otimes \mathcal{O}(2)$.

Then, the desired hol. 2-form on $X$ is given by $\Omega = \frac{d\tau}{y}$.

Another way to view this: use homo. coords $(u_1, u_2)$ for $\mathbb{CP}^4$; then we have five variables, i.e. $\mathbb{C}^5$, acted on by $\mathbb{C}^\times \times \mathbb{C}^\times$ with weights

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[$\Sigma = 12$]  
[$\Sigma = 3$]

The quotient defines a "weighted projective space" $W = \mathcal{O}_{\mathbb{C}^\times \times \mathbb{C}^\times}$. It has line bundles $\mathcal{O}(m+n)$ for $m, n \in \mathbb{Z}$. Canonical bundle $K = \mathcal{O}(-12, -3)$.

(The sums of the $\mathbb{C}^\times$-weights above) [Pf: use Euler seq. as for $\mathbb{CP}^4$]

And we defined $X$ as the vanishing locus of a section of $\mathcal{O}(12, 3)$.

$\implies$ Using normal bundle seq. as before, get $c_1(X) = 0$ as desired.

So, we've built a $K3$ surface as a fibers fibration, $\pi: X \to \mathbb{CP}^4$.

But the tori $\pi^{-1}(u)$ are complex, not real!

On the other hand, they are Lagrangian for $\Omega$.
(In loc. coord on $\mathbb{CP}^4$, $\Omega = du \wedge \frac{dx}{y}$, and du pulls back to 0 on $\pi^{-1}(u)$)

Or, better said: we have 3 natural real 2-forms in the sky,

$\omega_1 = \text{Re } \Omega$

$\omega_2 = \text{Im } \Omega$

$\omega_3 = \omega$  (Kähler form — for the Ricci-flat metric promised by Yau's Thm)
A special Lag. (w/ θ=0) has $i^*w_2 = i^*w_3 = 0$.
A complex Lag. has $i^*w_1 = i^*w_2 = 0$.

To turn one into the other, want to somehow permute the roles of $w_1, w_2, w_3$.

How?