Riemannian Geometry: Exercise Set 2

Exercise 1

Solve Exercises 3-3, 3-4, 3-7, 3-8 of Lee.

Exercise 2

Let \((V, \langle , \rangle)\) be a real vector space with positive definite inner product. As discussed in class, \(V\) is canonically a Riemannian manifold.

1. Let \(O(V)\) be the group of linear transformations of \(V\) preserving \(\langle , \rangle\). Show that \(O(V) \subset \text{Isom}(V)\).
2. For \(v \in V\) define the translation \(\varphi_v : V \to V\) by \(\varphi_v(w) = w + v\). Show that \(\varphi_v \in \text{Isom}(V)\), and the map \(v \mapsto \varphi_v\) embeds \(V\) as an abelian subgroup of \(\text{Isom}(V)\).
3. Show that \(\text{Isom}(V)\) contains the semidirect product of \(V\) and \(O(V)\), with \(O(V)\) acting on \(V\) in the obvious way. (Later we will prove that \(\text{Isom}(V)\) equals this semidirect product.)

Exercise 3

1. Let \((M, g) \subset (\tilde{M}, \tilde{g})\) be an embedded Riemannian submanifold. Suppose \(h \in \text{Isom}(\tilde{M}, \tilde{g})\) restricts to a map \(M \to M\). Show that \(h \in \text{Isom}(M, g)\).
2. Let \((V, \langle , \rangle)\) be a real vector space with positive definite inner product. Let \(S(V) = \{v \in V : \|v\| = 1\}\). Show that \(O(V) \subset \text{Isom}(S(V))\).

Exercise 4

Let \((M, g)\) be a Riemannian manifold. Let \(\langle , \rangle\) denote the inner product on any \(T^k(M)\) induced by \(g\).

1. Suppose \(\eta, \omega \in T^1(M)\). Show that \(\langle \eta^k, \omega^k \rangle = \langle \eta, \omega \rangle\).

Exercise 5

Let \((M, g)\) be a Riemannian manifold and \(f \in C^\infty(M)\).

1. Show that \(\text{grad } f\) points in the direction of fastest increase of the function \(f\) (part of the exercise is to figure out precisely what this sentence should mean).