In diff. top. you have studied the notion of smooth manifold. It's something that looks "locally like $\mathbb{R}^n".

In particular, locally any two smooth manifolds look identical. So e.g. by "local measurements" using only the structure of smooth mfd, can't distinguish $\mathbb{R}^2$ from $S^2$.

This doesn't capture many of the features of manifolds we see in the real world.

In particular: suppose you are an ant living on some general manifold $M \subset \mathbb{R}^3$.

Then, you could distinguish $M$ from $\mathbb{R}^2$. How?

Define

Length of a path: $Y: [0, T] \rightarrow M \subset \mathbb{R}^3$  \[ L(Y) = \int_0^T \|\dot{Y}\| dt \quad \text{with} \quad \|V\| = \sqrt{V \cdot V} \]

Define angle between two paths: $\sim \dot{X}, \dot{Y}$  \[ \dot{X} \cdot \dot{Y} = \|X\| \|Y\| \cos \Theta \]

Define geodesics to be paths on $M$ which locally minimize distance between two points.

Then, study geodesic triangles on $M$. \( \triangle \) Let $C$ be the sum of the interior angles.

You will find that $C \neq \pi$ in general.

For example: if $M = S^2$ of radius $R$, for a triangle of area $A$, find $C = \pi + \frac{A}{R^2}$

In general we may define $S(p) = \lim_{A \rightarrow 0} (C - \pi)/A$ \( \triangle \) ("scalar curvature").

Then

\[
S(p) = \begin{cases} 
\frac{1}{R^2} & \text{if } M = S^2 \text{ of radius } R \\
0 & \text{if } M = \mathbb{R}^2
\end{cases}
\]

So this is a local invariant of $M \subset \mathbb{R}^3$. We defined it using the notion of length and angle inherited from $\mathbb{R}^2$. 

\[ \text{Intro} \]
Amazing fact [Gauss-Bonnet]:

\[ \chi(M) = \frac{1}{4\pi} \int_M S \, dA \]

\[ \chi = 2 \]
\[ \chi = 0 \]
\[ \chi = -2 \]

so the ants living on \( M \) can determine its global topology just by making local measurements!

---

Our main aim in this course is to understand this "curvature" and its higher-dimensional analogues. For this we will need to study manifolds equipped with notions of dot-product of tangent vectors, \( g(x): T_xM \otimes T_xM \to \mathbb{R} \) symmetric positive definite.

aka, Riemannian metrics. [NB, this isn't the only possible notion: more generally, one could have just a norm \( F(x): T_xM \to \mathbb{R} \), "Finsler metric"]

In the example we just discussed, \( g \) was inherited: \( TM \subset T\mathbb{R}^3 \)

\( T\mathbb{R}^3 \) has standard metric \( g \),

\[ g_M = g_{can} \big|_{T_xM} \]

But in many cases \( M \) will not be a submanifold of anything — define \( g \) in some other way.

Curvature will turn out to be a 4-tensor \( R \in T^4_1(M) \), subject to constraints

so that it has \( \frac{1}{12} n^2(n^2-1) \) independent components, e.g.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( n = 1 )</th>
<th>( n = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 2 )</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>
| \( n = 4 \) | 20 | ...

The information in this tensor is contained equivalently in sectional curvature

\[ K: \text{Gr}_2(TM) \to \mathbb{R} \] (roughly, curvature of "2-plane sections" of \( M \)).

---

Why study Riemannian metrics?

1) Many natural \( M \) come with natural \( g \).

2) Studying \( g \) sometimes gives info about \( M \): e.g.

a) Gauss-Bonnet + Chern: \( M \) cpt \( \Rightarrow \chi(M) = \frac{1}{(2\pi)^n} \int_M \text{Pf}(R) \) \( M \) \( \rightarrow \) an \( n \)-form built algebraically from \( R \).

b) Cartan-Hadamard: if \( M \) is simply connected and admits a metric \( g \) with all sectional curvatures \( \leq 0 \), then \( M \) is diffeo to \( \mathbb{R}^n \).
c) Hodge: \( g \) determines Laplacian operators \( \Delta_k: \Omega^k(M, \mathbb{R}) \to \Omega^k(M, \mathbb{R}) \), and we have canonically \( \ker \Delta_k \cong H^k(M, \mathbb{R}) \).

d) Riem. metrics are the key tool in Perelman’s pf of Poincaré conjecture:
\( M \) simply connected compact, \( \dim M = 3 \implies M \text{ is homeomorphic to } S^3 \).

3) Riem geometry (or a very slight generalization, semi-Riem geometry, where we have the reg. of positive definiteness) occurs in nature: indeed spacetime is a semi-Riem manifold, and its curvature is responsible for gravity!

Hope to be able to say something about all of these topics.

---

**Prereq:** rudiments of differential topology

- smooth manifold
- vector bundle
- tangent, cotangent, tensor bundles
- Lie derivative

[Sec 2 of Lee has a very brief review.]