Connections

Want a useful notion of "straight line" in a Riem manfd \((M,g)\).

**Fact:** \(Y: [0,T] \rightarrow \mathbb{R}^n\) is a straight line iff \(\exists T: [0,T] \rightarrow [0,T]\) s.t. \(\tilde{Y} = Y \circ T\) has \(\frac{d}{dt}(\tilde{Y}) = 0\), i.e. \(\tilde{Y} = \text{const}\)

How to make sense of this eq. on a general \((M,g)\) ? It makes no sense a priori since it involves comparing vectors in different tangent spaces. Nevertheless we want to define it.

**Intuition:** suppose \(i: M \hookrightarrow \mathbb{R}^n\) submanifold, \(Y: [0,T] \rightarrow M\), and \(X \in T(Y \times TM)\) i.e. \(X(t) \in T_{Y(t)} M\). Then have \(dY: T_p M \rightarrow T_{Y(t)} \mathbb{R}^n \cong \mathbb{R}^n\)

\[\Pi: \mathbb{R}^n \cong T_p \mathbb{R}^n \rightarrow T_p M\quad \text{orth proj.}\]

Then we could define \(D_t X = \Pi(\frac{d}{dt} X)\).

**Def** \(\text{M smooth mfd, } E \rightarrow M\) real vector bundle.

A connection on \(E\) is, for each open set \(U \subset M\), a map \(\nabla: E(E|_U) \rightarrow E(E|_U \otimes T^*U)\)

- compatible with restriction of \(U\) i.e. \(\nabla(U)(s)|_U = \nabla(U)(s|_U)\)
- \(\mathbb{R}\)-linear i.e. \(\nabla(cs + c's') = c\nabla(s) + c'\nabla(s')\)

(i.e. \(\nabla\) is a map of sheaves of \(\mathbb{R}\)-modules)

- Leibniz i.e. \(\nabla(fs) = (df)s + f\nabla s\)

Notation: usually write \(\nabla_X s\) for \(\nabla_X s\). ("covariant derivative in the direction \(X\")

In local coordinates we write \(\nabla_i = \frac{\partial}{\partial x^i}\).
Suppose we fix a local basis of sections \( \{ e_a \}_{a=1}^r \) for \( E \) (but not a local coordinate.)
Define "connection coeff." \( A^b_a \in T'(M) \) by \( \nabla e_a = A^b_a e_b \).

Expanding a general section as \( s = s^a e_a \) we have
\[
\nabla s = (ds^a)e_a + s^a \nabla e_a = (ds^a)e_a + s^a A^b_a e_b = (ds^a + A^b_a s^b) e_a
\]
i.e. \( (\nabla s)^a = ds^a + A^b_a s^b \)

It's tempting to organize \( A^b_a \) into \( A = A^a_b e_a \otimes e^b \in T' \otimes \text{End } E \).

But, consider what happens under a change of basis, \( e_a' = C^a_{a'} e_{a'} \).

We'll then have
\[
\nabla e_{a'} = \nabla (C^a_{a'} e_a) = dC^a_{a'} e_{a'} + C^a_{a'} A^b_a e_b
\]
\[= (C^{-1})^b_{a'} dC^a_{a'} e_{a'} + C^a_{a'} A^b_a (C^{-1})^b_{a'} e_{a'} \]
i.e. \( A^b_a' = (C^{-1})^b_{a'} A^a_{b'} C^a_{a'} + (C^{-1})^b_{a'} dC^a_{a'} \).

The second term reflects the fact that a connection is not a global section of any vector bundle over \( M \).

However, what is true is that the difference of two connections is a global \( \text{End}(E) \)-valued 1-form: i.e.
\[
(\nabla - \tilde{\nabla}) s = \omega \cdot s \quad \omega \in \mathcal{E}(T' \otimes \text{End}(E))
\]

[One way to understand this: \( A \) and \( \tilde{A} \) both transform by the same inhom. term, so \( A - \tilde{A} \) is indep. of basis. Another way: \( \nabla - \tilde{\nabla} \) is linear over \( C^\infty \)-functions, by Leibniz rule.]

Lemma: \( M \) smooth, \( E \) v.l.
\[ \Rightarrow \] the space of connections on \( E \) is an affine space modeled on \( \mathcal{E}(T' \otimes \text{End}(E)) \).

Pf: Just need to show a connection exists; use partition of unity.
Given \((\tilde{M}, E, \nabla)\) and \(\mathcal{Y}: M \rightarrow \tilde{M}\), there is pullback connection \(\mathcal{Y}^*\nabla\) on \(\mathcal{Y}^*E\), as follows:

Fix a local basis \(\{e_a\}\) of \(E\), then write the basis \(\{\mathcal{Y}^*e_a\}\) for \(\mathcal{Y}^*E\).

Define \(\mathcal{Y}^*\nabla\) by fixing its connection coeff to be \(\mathcal{Y}^*A^a_b\).

Then check that \(\mathcal{Y}^*\nabla\) is indep of the chosen basis (basically \(\mathcal{Y}^*(\text{C}^1\text{dC}) = \mathcal{Y}^*(\text{C}^1)\text{d}\mathcal{Y}^*(\text{C})\)).

\[\mathcal{Y}^*\nabla\text{ can also be characterized by the confusing equation:}\]

\[(\mathcal{Y}^*\nabla)_x(\mathcal{Y}^*s) = \mathcal{Y}^*(\nabla_{\mathcal{Y}_x^*(\mathcal{Y})(s)})\]

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**Lemma**

Fix \(\mathcal{Y}: [0,1] \rightarrow M\) and \(s_0 \in E_{\mathcal{Y}(0)}\).

Then \(\exists!\ s \in \mathcal{E}(\mathcal{Y}^*E)\ s.t. (\mathcal{Y}^*\nabla)_t s = 0,\ s(0) = s_0\).

**Pf**

Show that the interval on which \(s\) exists is:

- open — by theor of linear 1st-order ODE in a coord chart

\[\frac{ds^a}{dt} = -c^a_b(t)s^b, \text{ where } (c^a_b\text{d}t) \text{ are the connection form for } \mathcal{Y}^*\nabla\]

or explicitly, \(c^a_b = [A^a_b(\mathcal{Y})]\)

- closed — by taking a chart containing the limit point

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**Def**

(Parallel transport) \(P_{\mathcal{Y}}(s_0) = s(1)\).

**Lemma**

- \(P_{\mathcal{Y}}\) is an isomorphism \(E_{\mathcal{Y}(0)} \rightarrow E_{\mathcal{Y}(1)}\).

- If \(\mathcal{Y} = \mathcal{Y}_1, \mathcal{Y}_2\) then \(P_{\mathcal{Y}_1} \circ P_{\mathcal{Y}_2} = P_{\mathcal{Y}}\).

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**Operations on connections**

- Given \(\nabla_1, \nabla_2\) in \(E_1, E_2\) get \(\nabla\) on \(E_1 \oplus E_2\) by \(\nabla(s_1, s_2) = (\nabla_1s_1, \nabla_2s_2)\)

- Given \(\nabla\) on \(E\) get \(\nabla\) on \(E^*\) by \(\text{d}(\omega)_s = (\nabla_\omega)s + \omega(\nabla_s)\)