Andy class.

$U \subset \mathbb{R}^n$ open.
$x_i : x_i (t) \in \mathbb{R}$

$q_{ij} : U \rightarrow \mathbb{R}$ metric: $g = q_{ij} dx_idx_j$

$x : [0, T] \rightarrow U$ smooth. $x(t) = (x_1(t), \ldots, x_n(t))$

$L(x) = \int_0^T dt \sqrt{g_{ij}(x(t)) \dot{x}^i(t) \dot{x}^j(t)}$

Rank: $L(x)$ is reparameterization invariant: $t \rightarrow t' \in [0, T]$

$= \int_0^{T'} dt' \sqrt{g_{ij}(x(t')) \frac{dx^i(t')}{dt'} \frac{dx^j(t')}{dt'}}$

Prop: If $\delta(t) \rightarrow 0$ for all $t$, then there is a unique $\theta$ s.t. $\left| \frac{d}{dt}, x(t) \right| = 1$.

Assume so for a given parameterization. Then $L(\theta) = T$.

Now differentiate length.

$\theta : [0, T] \times (-\epsilon, \epsilon) \rightarrow U$ smooth. $\theta(t) = P(t, s)$

$\theta(t, 0) = x(t)$ parameterized by arc length.

$\left. \frac{d}{ds} \right|_{s=0} L(\theta)$ is variation of arc length.

$\frac{d}{ds} |_{s=0}$
\[
\frac{\partial}{\partial s} \left. \frac{x_s}{s=0} \right|_t = \left[ \Theta_s \right]_t = \sum_t \sqrt{g_{ij}(x(t, s)) \partial x^i(t, s) \partial x^j(t, s)}
\]

\[
= \int_0^T dt \left[ \frac{\partial g_{ij}(x)}{\partial x^k} \frac{\partial x^i}{\partial x^j} \frac{\partial x^k}{\partial x^j} + g_{ij}(x) \frac{\partial^2 x^i}{\partial x^j} \right]
\]

\[
= \int_0^T dt \left[ \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \frac{\partial x^i}{\partial x^j} \frac{\partial x^k}{\partial x^j} + \partial_t \left( \frac{\partial g_{ij}(x)}{\partial x^k} \frac{\partial x^i}{\partial x^j} \right) - \frac{\partial g_{ij}(x)}{\partial x^k} \frac{\partial x^k}{\partial x^j} \frac{\partial x^j}{\partial x^i} - g_{ij}(x) \frac{\partial^2 x^i}{\partial x^j} \right]
\]

\[
= \int_0^T dt \left[ \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \frac{\partial x^i}{\partial x^j} \frac{\partial x^k}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \frac{\partial x^i}{\partial x^j} \frac{\partial x^k}{\partial x^j} - g_{ij}(x) \frac{\partial^2 x^i}{\partial x^j} \right]
\]

\[
+ g_{ij}(x) \frac{\partial x^i}{\partial x^j} \frac{\partial x^i}{\partial x^j} \left[ \Theta_i \right]_t^T
\]
If to be zero,

\[ g_{kj} \dot{x}^j + \frac{\partial g_{kj}}{\partial x^i} \dot{x}^i \dot{x}^j - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j = 0 \quad \text{for all } k. \]

Remark:

\[ \left[ g_{kj} \frac{\partial}{\partial t} + \frac{1}{2} \left( \frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right) \dot{x}^i \right] \dot{x}^j = 0 \quad \forall \lambda \]

So

\[ \frac{\partial x^l}{\partial t} + \Gamma_{ij} \dot{x}^i \dot{x}^j = 0 \quad \forall \lambda \]  \hspace{1cm} (\star)

Define covariant derivative operator on \( E = \mathcal{U} \times \mathbb{R}^n \) by

\[ \nabla_{\partial x^i} \frac{\partial x^j}{\partial x^i} = \Gamma_{ij} \frac{\partial}{\partial x^j}, \quad \Gamma_{ij} : \mathcal{U} \to \mathbb{R} \]

So (\star) is

\[ \nabla_{\partial x^i} \left( \frac{\partial x^j}{\partial x^i} \right) = \dot{x}^i \frac{\partial x^1}{\partial x^i} \frac{\partial}{\partial x^j} + \dot{x}^i \dot{x}^j \Gamma_{ij} \frac{\partial}{\partial x^j} \]

Explain that company for \( [0, T] \to \mathbb{R}^n \), so pullback \( \nabla \)
So if $x : [0, T] \rightarrow \mathbb{R}^n$, then $\nabla_x \dot{x} = 0$.

**Interpretation:**
- Zero acceleration
- $\dot{x}$ only defined along $x$ but integrates on $x^* T^* \mathbb{R}^n$.
- Pullback covariant derivative.

**Intrinsic approach:**

1. $E, \langle \cdot, \cdot \rangle$ metric
   \[
   \nabla \text{ symmetric if }\n   \forall <s_1, s_2> = <\nabla s_1, s_2> + <s_1, \nabla s_2> \text{ for all sections } s_1, s_2.
   \]
   Equivalent: $g : E \otimes E \rightarrow \mathbb{R}$
   \[
   \nabla g = 0.
   \]

2. $TM$, $\nabla$ covariant derivative
   \[
   \text{Def: Torsion } \tau(\xi, \xi) = \nabla_{\xi_1} \xi_2 - \nabla_{\xi_2} \xi_1 - [\xi_1, \xi_2].
   \]
   $\tau : \Lambda^2 TM \rightarrow TM$ is a tensor.

(Reci, his teacher)

(1917)

(Levi-Civita): $M$ Riemannian. \exists $\nabla$ covariant, torsion-free.

Ref: Capture $\langle \nabla_{\frac{\partial}{\partial s_1}} \frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_3} \rangle$.

Mysterious fundamental part of Riemannian geometry!
In 1900 he and Ricci-Curbastro published the theory of tensors in *Méthodes de calcul différentiel absolu et leurs applications*, which Albert Einstein used as a resource to master the tensor calculus, a critical tool in Einstein's development of the theory of general relativity. Levi-Civita's series of papers on the problem of a static gravitational field were also discussed in his 1915–1917 correspondence with Einstein. The correspondence was initiated by Levi-Civita, as he found mathematical errors in Einstein's use of tensor calculus to explain theory of relativity. Levi-Civita methodically kept all of Einstein's replies to him, and even though Einstein hadn't kept Levi-Civita's, the entire correspondence could be reconstructed from Levi-Civita's archive. It's evident from these letters that, after numerous letters, the two men had grown to respect each other. In one of the letters, regarding Levi-Civita's new work, Einstein wrote "I admire the elegance of your method of computation; it must be nice to ride through these fields upon the horse of true mathematics while the like of us have to make our way laboriously on foot". In 1933 Levi-Civita contributed to Paul Dirac's equations in quantum mechanics as well.[6]
In cad system derive \( \nabla_{\tilde{\tau}}^2 \psi = \frac{\partial^2 \psi}{\partial x^2} \).

Finally, revisit computation:

\[ \gamma: [0, T] \rightarrow M \]

\[ \Gamma: [0, T] \times (-\epsilon, \epsilon) \rightarrow M \]

Capture on \( P \).

So

\[ \tilde{T} = \frac{\partial \gamma}{\partial t} \quad \tilde{s} = \frac{\partial \gamma}{\partial s} \]

of \( P^* TM \rightarrow P \). So

\[ L = \int_0^T dt \left< \tilde{T}, \tilde{s} \right> \]

(evaluated at \( s \)).

\[ \partial_s L = \partial_s \int_0^T dt \left< \tilde{T}, \tilde{s} \right> \]

\[ = \int_0^T dt \left. \partial_s \left< \tilde{T}, \tilde{s} \right> \right|_{s=0} \]

Orthogonal

\[ = \int_0^T dt \left. \partial_s \left< \tilde{\tau}, \tilde{\tau} \right> \right|_{s=0} \]

Face free, \( < \tilde{\tau}, \tilde{\tau} > = 1 \) at \( s = 0 \)

Orthogonal

\[ = \int_0^T dt \left( \tilde{\tau} \left< \tilde{\tau}, \tilde{\tau} \right> - \left< \tilde{\tau}, \nabla_\tilde{\tau} \tilde{\tau} \right> \right|_{s=0} \]

Orthogonal

\[ = \left< \tilde{\tau}, \tilde{\tau} \right> \int_0^T dt - \int_0^T \left< \tilde{\tau}, \nabla_\tilde{\tau} \tilde{\tau} \right> dt \]

see again gradient equation \( \nabla_\tilde{\tau} \tilde{\tau} = 0 \).