More on Levi-Civita connection and geodesics

Last time: \((\text{Mg})\) Riem manifold

\[ \Rightarrow \exists! \text{ orthogonal, torsion-free connection } \nabla \text{ on } TM. \]

In local coords, \( \nabla^2 \left( \frac{2}{2x^i} \right) = \Gamma^k_{ij} \frac{\partial}{\partial x^k} \) \( (\Gamma^k_{ij} dx^i = A^k_j \text{ connection coeff}) \)

\[ \Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \]

**Prop** The induced connection on a submanifold \( M \subset \mathbb{R}^n \) (defined using ortho. projection) coincides with the Levi-Civita connection. [Exercise]

**Prop** If \( \Phi : M \to \tilde{M} \) is a isometry and \( \nabla, \tilde{\nabla} \) are Levi-Civita then \( \nabla = \Phi^* \tilde{\nabla} \).

**Pf** Just show \( \Phi^* \tilde{\nabla} \) is orthogonal, torsion-free.

**Def** \( \gamma : [0, T] \to M \) is geodesic if \( (\Phi^* \tilde{\nabla})_2 (\gamma^* \dot{\gamma}) = 0 \), i.e. \( \ddot{\gamma} + \Gamma^j_{ik} \dot{\gamma}^i \dot{\gamma}^k = 0 \)

(Sometimes also written \( \nabla_{\dot{\gamma}} \dot{\gamma} = 0 \), using the fact that this will be independent of how we extend \( \dot{\gamma} \) to the full \( M \))

**Prop** Geodesics are local extrema of length (under variation with fixed endpoints)

**Pf** Computation in last lecture. \( \left[ \frac{d}{ds} \right]_{s=0} L(\gamma_s) = - \int_0^T \left< \dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma} \right> dt \)

**Prop** For any \( v \in T_p M \), there exists \( \varepsilon > 0 \) and a geodesic \( \gamma : [0, \varepsilon] \to M \) with \( \gamma(0) = p, \dot{\gamma}(0) = v \)

Any two such geodesics agree on their common domain.

**Pf** Existence + uniqueness theorem for nonlinear 1st-order DE.

Let \( \gamma_v \) denote the geodesic \( \gamma_v : [0, T] \to M \) with \( \gamma_v(0) = p, \dot{\gamma}_v(0) = v \), and \( T \) maximal.

(maybe \( T = \infty \)).

**Prop** For any \( v \in T_p M \) and \( c, t \in \mathbb{R} \), \( \gamma_{cv}(t) = \gamma_v(ct) \).

**Pf** Just check \( t \mapsto \gamma_v(ct) \) is indeed a geodesic w/ initial vector \( cv \).
Def Let $D = \{ v \in TM : \gamma_v(t) \text{ is defined at } t = 1 \}$.

Exponential map is $\exp: D \to M$

$$v \mapsto \gamma_v(1)$$

Prop

a) $D \subset TM$ is open, each $D_p \subset T_pM$ star-shaped

b) $\gamma_v(t) = \exp(tv)$ when both sides defined

c) $\exp$ is smooth.

Pf

b) is easy. For a), c): interpret geodesics as integral curves of a global vector field $G$ on $TM$.

local $x^i$ on $M \Rightarrow$ coords $(x^i, v^i)$ on $TM$:

$$G = v^k \frac{\partial}{\partial x^k} - v^i \Gamma^k_{ij}(x) \frac{\partial}{\partial v^k}$$

Then flow by $G$ is $\frac{dx^k}{dt} = G(x^k) = v^k$, $\frac{dv^k}{dt} = G(v^k) = -v^i \Gamma^k_{ij}(x)$.

So $\exp$ is the time-1 flow of $G$, composed with $\iota_T: TM \to M$.

But time-1 flow of a smooth vector field is smooth (ODE: $t \mapsto \gamma_t(x)$ is $C^\infty$). The locus where time-1 flow exists is open and $\iota_T$ is an open mapping.

I sometimes take geodesics to geodesics, i.e. $\Phi: M \to \tilde{M}$ is $\Rightarrow \exp(\Phi \ast tv) = \Phi(\exp tv)$

Use $\Phi \ast \nabla = \nabla$.

Ex Geodesics in $\mathbb{R}^n$ are straight lines. (Pf $\Gamma^k_{ij} = 0$ so geodesic eq becomes $\ddot{x}^i = 0$)

$$\gamma_v(t) = p + tv \quad \exp((p, v)) = p + v \quad D = TM$$

Ex Geodesics in $S^n$ are great circles, i.e. intersections between $S^n$ and 2-planes in $\mathbb{R}^{n+1}$.

(e.g. geodesics thru south pole are straight lines thru $O$ in the stereographic coordinate $u$.)

Pf Could compute directly using formula for $T$.

But, easier to use isometry group $O(n+1)$.

In $\mathbb{R}^n$, consider $p = (1, 0, \ldots, 0)$ $v = (0, 1, \ldots, 0)$

There is an isometry $\Psi: (x^1, \ldots, x^n) \mapsto (x_1, x_2, -x_3, \ldots, -x_{n+1})$ with $\Psi \ast v = v$.

Thus $\Psi(\exp tv) = \exp t \Psi \ast v = \exp tv$.

So $\exp tv$ lies in $S^n \cap \mathbb{R}^2$. From here it's easy to see it is a great circle with unit speed parameterization.

Get all other geodesics by acting on this one by isometries.
Ex Geodesics in $H^n$ are great hyperbolas, i.e. intersections between $H^n$ and 2-planes in $T^n$. 

Pf Similar to above.