Riemannian distance

$$(M, g) \text{ Riem manifold } p, p' \in M$$

Def: an admissible curve in $M$ is a map $\gamma: [0, T] \to M$ which is piecewise immersion (divided into finitely many segments; smooth w/ $\dot{\gamma} \neq 0$ on each segment).

Def: length $L(\gamma)$ of an admissible curve is $\Sigma$ of lengths of the segments $\int_a^b \| \dot{\gamma}(t) \| dt$.

Def: For $p, p' \in M$, $d(p, p') = \inf \{ L(\gamma) : \gamma \text{ admissible with } \gamma(0) = p, \gamma(T) = p' \}$.

Lemma: $d(p, p')$ makes $M$ a metric space, inducing the usual topology on $M$.

Proof: $d(p, p') \geq 0$ easy.

$\Delta$ inequ. easy.

Need to show that $d(p, p') > 0$ for $p \neq p'$.

For this, pick a ball $U$ of $p$, and curves $\gamma_j(p) : \mathbb{R} \to M$.

Shrink $U$ if necessary we can arrange $p \notin U$.

Let $K = \{ (\gamma, v) \in TM : v \in \mathbb{R}^m, \| v \| = 1 \}$. $K$ compact. Then define $c = \min_{\gamma \in K} \| v \|_{\gamma}$.

Then $\| v \|_{\gamma} > c \| v \|_{\gamma}$ where $v_j = e_j$. U contains some coordinate $\varepsilon$-ball around $p$.

Thus any path which exits $U$ has length $\geq \varepsilon$ in the metric $h_i$.

Thus, $\int_0^T \sqrt{\gamma''(t)^2} dt \geq c \varepsilon$.

So $d(p, p') > c \varepsilon$.

To compare topologies, fix a coordinate ball $B_{\varepsilon}(p) = B_{\varepsilon, h}(p)$. On this ball $c \| v \|_{\gamma} < \| v \|_{\gamma}$.

So $B_{\varepsilon, h}(p) \supset B_{\varepsilon, g}(p)$.

Thus, $L(\gamma) < c \varepsilon \Rightarrow L_h(\gamma) < \frac{1}{c} L_g(\gamma) < \varepsilon$. 

Similarly in the other direction.